

Simple Toroidal Vertex Algebras and Their Irreducible Modules

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Abstract

In this paper, we continue the study on toroidal vertex algebras initiated in [LTW1], to study concrete toroidal vertex algebras associated to toroidal Lie algebra $L_r(\hat{\mathfrak{g}}) = \hat{\mathfrak{g}} \otimes L_r$, where $\hat{\mathfrak{g}}$ is an untwisted affine Lie algebra and $L_r = \mathbb{C}[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$. We first construct an $(r+1)$ -toroidal vertex algebra $V(T, 0)$ and show that the category of restricted $L_r(\hat{\mathfrak{g}})$ -modules is canonically isomorphic to that of $V(T, 0)$ -modules. Let \mathfrak{c} denote the standard central element of $\hat{\mathfrak{g}}$ and set $S_{\mathfrak{c}} = U(L_r(\mathbb{C}\mathfrak{c}))$. We furthermore study a distinguished subalgebra of $V(T, 0)$, denoted by $V(S_{\mathfrak{c}}, 0)$. We show that (graded) simple quotient toroidal vertex algebras of $V(S_{\mathfrak{c}}, 0)$ are parametrized by a \mathbb{Z}^r -graded ring homomorphism $\psi : S_{\mathfrak{c}} \rightarrow L_r$ such that $\text{Im}\psi$ is a \mathbb{Z}^r -graded simple $S_{\mathfrak{c}}$ -module. Denote by $L(\psi, 0)$ the simple quotient $(r+1)$ -toroidal vertex algebra of $V(S_{\mathfrak{c}}, 0)$ associated to ψ . We determine for which ψ , $L(\psi, 0)$ is an integrable $L_r(\hat{\mathfrak{g}})$ -module and we then classify irreducible $L(\psi, 0)$ -modules for such a ψ . For our need, we also obtain various general results.

1 Introduction

Let \mathfrak{g} be a finite-dimensional simple Lie algebra equipped with the normalized Killing form $\langle \cdot, \cdot \rangle$. Let $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t_0^{\pm 1}] \oplus \mathbb{C}\mathfrak{c}$ be the untwisted affine Lie algebra. It is well-known (see [FZ], [Li1]) that there exists a canonical vertex algebra $V_{\hat{\mathfrak{g}}}(\ell, 0)$ associated to $\hat{\mathfrak{g}}$ for each $\ell \in \mathbb{C}$ and the category of $V_{\hat{\mathfrak{g}}}(\ell, 0)$ -modules is canonically isomorphic to the category of restricted $\hat{\mathfrak{g}}$ -modules of level ℓ . Denote by $L_{\hat{\mathfrak{g}}}(\ell, 0)$ the unique graded simple quotient vertex algebra of $V_{\hat{\mathfrak{g}}}(\ell, 0)$. It was known (see [K]) that $L_{\hat{\mathfrak{g}}}(\ell, 0)$ is an integrable $\hat{\mathfrak{g}}$ -module if and only if ℓ is a non-negative integer. Furthermore, it was known (see [FZ], [DL], [Li1], [MP1], [MP2], [DLM]) that if ℓ is a non-negative integer, the category of $L_{\hat{\mathfrak{g}}}(\ell, 0)$ -modules is naturally isomorphic to the category of restricted integrable $\hat{\mathfrak{g}}$ -modules of level ℓ .

Toroidal Lie algebras, which are essentially central extensions of multi-loop Lie algebras, generalizing affine Kac-Moody Lie algebras, form a special family of infinite dimensional Lie algebras closely related to extended affine Lie algebras (see

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[AABGP]). A natural connection of toroidal Lie algebras with vertex algebras has also been known (see [BBS]), which uses one-variable generating functions for toroidal Lie algebras. By considering multi-variable generating functions (cf. [IKU], [IKUX]), toroidal vertex algebras were introduced in [LTW1], which generalize vertex algebras in a certain natural way.

The essence of an $(r+1)$ -toroidal vertex algebra V is that to each vector $v \in V$, a multi-variable vertex operator $Y(v; x_0, \mathbf{x})$ is associated, which satisfies a Jacobi identity. It is important to note that for a vertex algebra $(V, Y, \mathbf{1})$, the so-called creation property states that

$$Y(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \quad (Y(v, x)\mathbf{1})|_{x=0} = v \quad \text{for } v \in V,$$

which implies that V as a V -module is cyclic on the vacuum vector $\mathbf{1}$ and the vertex operator map $Y(\cdot, x)$ is always injective. However, this is *not* the case for an $(r+1)$ -toroidal vertex algebra in general. For an $(r+1)$ -toroidal vertex algebra V , denote by V^0 the submodule of the adjoint module V generated by $\mathbf{1}$, which is an $(r+1)$ -toroidal vertex subalgebra. It was proved in [LTW1] that V^0 has a canonical vertex algebra structure. To a certain extent, V^0 to V is the same as the core subalgebra to an extended affine Lie algebra (see [AABGP]). In this paper, we explore V^0 more in various directions. In particular, we show that V^0 is a vertex \mathbb{Z}^r -graded algebra in a certain sense (see Section 3 for the definition). It is proved that if V is a simple $(r+1)$ -toroidal vertex algebra, then V^0 is also a simple $(r+1)$ -toroidal vertex algebra. Let L be any quotient $(r+1)$ -toroidal vertex algebra of V . It is proved (see Proposition 2.26) that a V -module W is naturally an L -module if and only if W is naturally an L^0 -module.

In this paper, we also study $(r+1)$ -toroidal vertex algebras naturally arisen from toroidal Lie algebras. Specifically, we consider Lie algebra

$$\tau = \hat{\mathfrak{g}} \otimes \mathbb{C} [t_1^{\pm 1}, \dots, t_r^{\pm 1}], \quad (1.1)$$

the r -loop algebra of an untwisted affine Lie algebra $\hat{\mathfrak{g}}$. We here study all possible $(r+1)$ -toroidal vertex algebras associated to τ . The most general $(r+1)$ -toroidal vertex algebra we get is $V(T, 0)$, whose underlying space is the induced τ -module

$$V(T, 0) = U(\tau) \otimes_{U(L_r(\mathfrak{g} \otimes \mathbb{C}[t_0]))} T, \quad (1.2)$$

where $T = \mathfrak{g} \oplus \mathbb{C}\mathfrak{c} \oplus \mathbb{C}$ is a certain $L_r(\mathfrak{g} \otimes \mathbb{C}[t_0])$ -module (see Lemma 4.3 and Theorem 4.4 for details). We establish an equivalence between the category of $V(T, 0)$ -modules and the category of restricted τ -modules (see Theorem 4.5).

An important $(r+1)$ -toroidal vertex algebra is the subalgebra $V(T, 0)^0$ of $V(T, 0)$, alternatively denoted by $V(S_{\mathfrak{c}}, 0)$, where $S_{\mathfrak{c}} = S \left(\bigoplus_{\mathbf{m} \in \mathbb{Z}^r} \mathbb{C}(\mathfrak{c} \otimes \mathfrak{t}^{\mathbf{m}}) \right)$, a subalgebra of $U(\tau)$. We furthermore study simple quotient $(r+1)$ -toroidal vertex algebras of $V(S_{\mathfrak{c}}, 0)$. Let $\psi : S_{\mathfrak{c}} \rightarrow L_r$ be a \mathbb{Z}^r -graded algebra homomorphism such that $\text{Im} \psi$ is a \mathbb{Z}^r -graded simple $S_{\mathfrak{c}}$ -module (following Rao [R1]), and let $I(\psi)$ be the ideal of

$V(S_c, 0)$ generated by $\text{Ker}\psi$. Denote by $V(\psi, 0)$ the quotient $(r+1)$ -toroidal vertex algebra of $V(S_c, 0)$ modulo $I(\psi)$. It is proved that $V(\psi, 0)$ has a unique graded simple quotient $(r+1)$ -toroidal vertex algebra, which is denoted by $L(\psi, 0)$, and these toroidal vertex algebras $L(\psi, 0)$ exhaust all graded simple quotient $(r+1)$ -toroidal vertex algebras of $V(S_c, 0)$. Among the main results, we prove that $L(\psi, 0)$ is an integrable τ -module if and only if ψ is given by

$$\psi(\mathbf{c} \otimes \mathbf{t}^{\mathbf{m}}) = \left(\sum_{i=1}^s \ell_i \mathbf{a}_i^{\mathbf{m}} \right) \mathbf{t}^{\mathbf{m}} \quad (\mathbf{m} \in \mathbb{Z}^r) \quad (1.3)$$

for some finitely many positive integers ℓ_1, \dots, ℓ_s and some distinct elements $\mathbf{a}_1, \dots, \mathbf{a}_s \in (\mathbb{C}^\times)^r$ ($\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$). Furthermore, assuming that $L(\psi, 0)$ is an integrable τ -module, we find a necessary and sufficient condition that a τ -module is an $L(\psi, 0)$ -module, and moreover, we classify all irreducible $L(\psi, 0)$ -modules.

For this paper, determining the integrability of $L(\psi, 0)$ and its irreducible modules is the core. One of the difficult problems is to find the explicit characterization of $L(\psi, 0)$ in terms of $V(\psi, 0)$. (Compared with the case for the ordinary affine vertex algebras, this problem is much harder.) To achieve these goals, we extensively use a graded simple quotient vertex algebra $L^0(\psi, 0)$ of $L(\psi, 0)$ and we show that $L(\psi, 0)$ can be embedded canonically into the tensor product vertex algebra $L^0(\psi, 0) \otimes L_r$, which is naturally a toroidal vertex algebra. As an important step, by using a result of Rao we show that for a ψ given by (1.3), $L^0(\psi, 0)$ is isomorphic to the tensor product of simple vertex algebras $L_{\hat{\mathfrak{g}}}(\ell_i, 0)$ for $i = 1, \dots, s$. Eventually, we achieve a classification of irreducible $L(\psi, 0)$ -modules in terms of irreducible $L_{\hat{\mathfrak{g}}}(\ell_i, 0)$ -modules.

We mention that the present work is closely related to some of Rao's. Let $\tilde{\tau}$ be the semi-direct product of τ and the degree derivations d_0, \dots, d_r . In [R2], Rao classified irreducible integrable $\tilde{\tau}$ -modules with finite dimensional weight spaces. It was implicitly proved therein that any irreducible integrable restricted $\tau \oplus \mathbb{C}d_0$ -module with finite dimensional weight spaces is isomorphic to a concrete module associated to some finitely many integrable highest weight $\hat{\mathfrak{g}}$ -modules and some distinct vectors in $(\mathbb{C}^\times)^r$ of the same number. When $L(\psi, 0)$ is an integrable τ -module, irreducible $L(\psi, 0)$ -modules are closely related to these τ -modules. Rao's results are very helpful at certain stages of this work.

This paper is organized as follows: In Section 2, after recall the basics about a general $(r+1)$ -toroidal vertex algebra V and its subalgebra V^0 , we study the connection between the module category of V and that of V^0 viewed as an $(r+1)$ -toroidal vertex algebra and as a vertex algebra. In Section 3, we study $(r+1)$ -toroidal vertex algebra $L_r(A) = A \otimes \mathbb{C}[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$ associated to a vertex algebra A and we relate an $(r+1)$ -toroidal vertex algebra V to $L_r(A)$ for some vertex algebra A . In Section 4, by using toroidal Lie algebra τ we construct and study $(r+1)$ -toroidal vertex algebras $V(T, 0)$, $V(S_c, 0)$, $V(\psi, 0)$, and $L(\psi, 0)$. In Section 5, we determine when $L(\psi, 0)$ viewed as a τ -module is integrable. In Section 6, we classify irreducible modules for $L(\psi, 0)$ with $L(\psi, 0)$ an integrable τ -module.

2 Toroidal Vertex Algebras and Vertex Algebras

In this section, we first recall some basic definitions and then study ideals of a toroidal vertex algebra, simple toroidal vertex algebras and their module categories.

We begin by recalling the notion of toroidal vertex algebra. Let r be a positive integer. For a vector space W , set

$$\mathcal{E}(W, r) = \text{Hom}(W, W[[x_1^{\pm 1}, \dots, x_r^{\pm 1}]][(x_0)])$$

and

$$\mathcal{E}(W) = \text{Hom}(W, W((x_0))).$$

Set $\mathbf{x} = (x_1, \dots, x_r)$. For $\mathbf{m} \in \mathbb{Z}^r$, set

$$\mathbf{x}^{\mathbf{m}} = x_1^{m_1} \cdots x_r^{m_r}.$$

The following notion was introduced in [LTW1]:

Definition 2.1. An $(r+1)$ -toroidal vertex algebra is a vector space V , equipped with a linear map

$$\begin{aligned} Y(\cdot; x_0, \mathbf{x}) : V &\rightarrow \mathcal{E}(V, r), \\ v &\mapsto Y(v; x_0, \mathbf{x}) = \sum_{(m_0, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r} v_{m_0, \mathbf{m}} x_0^{-m_0-1} \mathbf{x}^{-\mathbf{m}} \end{aligned}$$

and equipped with a vector $\mathbf{1} \in V$, satisfying the conditions that

$$Y(\mathbf{1}; x_0, \mathbf{x})v = v \quad \text{and} \quad Y(v; x_0, \mathbf{x})\mathbf{1} \in V[[x_0, x_1^{\pm 1}, \dots, x_r^{\pm 1}]] \quad \text{for } v \in V,$$

and that for $u, v \in V$,

$$\begin{aligned} z_0^{-1} \delta \left(\frac{x_0 - y_0}{z_0} \right) Y(u; x_0, \mathbf{zy}) Y(v; y_0, \mathbf{y}) &- z_0^{-1} \delta \left(\frac{y_0 - x_0}{-z_0} \right) Y(v; y_0, \mathbf{y}) Y(u; x_0, \mathbf{zy}) \\ &= y_0^{-1} \delta \left(\frac{x_0 - z_0}{y_0} \right) Y(Y(u; z_0, \mathbf{z})v; y_0, \mathbf{y}), \end{aligned} \tag{2.1}$$

where

$$Y(u; x_0, \mathbf{zy}) = \sum_{(m_0, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r} u_{m_0, \mathbf{m}} x_0^{-m_0-1} \mathbf{z}^{-\mathbf{m}} \mathbf{y}^{-\mathbf{m}}.$$

For a given $(r+1)$ -toroidal vertex algebra V , the notion of $(r+1)$ -toroidal vertex subalgebra and that of ideal are defined in the obvious way. On the other hand, the notion of V -module is also defined in the obvious way; a V -module is a vector space W equipped with a linear map $Y_W(\cdot; x_0, \mathbf{x})$ from V to $\mathcal{E}(W, r)$ such that

$$Y_W(\mathbf{1}; x_0, \mathbf{x}) = 1_W \quad (\text{the identity operator on } W)$$

and such that for any $u, v \in V$, the Jacobi identity (2.1) with Y replaced by Y_W in five obvious places holds.

From now on, we fix the positive integer r and we simply call an $(r+1)$ -toroidal vertex algebra a toroidal vertex algebra.

Definition 2.2. Let V be a toroidal vertex algebra. A *derivation* of V is a linear operator D on V , satisfying the condition that

$$D(\mathbf{1}) = 0 \quad \text{and} \quad [D, Y(v; x_0, \mathbf{x})] = Y(D(v); x_0, \mathbf{x}) \quad \text{for } v \in V. \quad (2.2)$$

Definition 2.3. An *extended toroidal vertex algebra* is a toroidal vertex algebra V equipped with derivations $\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_r$ such that

$$\begin{aligned} Y(\mathcal{D}_0(v); x_0, \mathbf{x}) &= \frac{\partial}{\partial x_0} Y(v; x_0, \mathbf{x}), \\ Y(\mathcal{D}_j(v); x_0, \mathbf{x}) &= \left(x_j \frac{\partial}{\partial x_j} \right) Y(v; x_0, \mathbf{x}) \end{aligned} \quad (2.3)$$

for $v \in V$, $1 \leq j \leq r$.

Remark 2.4. Note that in the definition of a vertex algebra V , the so-called creation property states that for every $v \in V$,

$$Y(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x \rightarrow 0} Y(v, x)\mathbf{1} = v.$$

It follows that the adjoint module V is cyclic on the vacuum vector $\mathbf{1}$ and the vertex operator map $Y(\cdot, x)$ of the vertex algebra is *always* injective. In contrast to this, for a toroidal vertex algebra V , while such a creation property is missing, V as a V -module may be not cyclic on $\mathbf{1}$, and the vertex operator map $Y(\cdot; x_0, \mathbf{x})$ may be not injective.

The following result was obtained in [LTW1]:

Proposition 2.5. *Let V be a toroidal vertex algebra. Set*

$$V^0 = \text{span}\{v_{m_0, \mathbf{m}}\mathbf{1} \mid v \in V, (m_0, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r\}. \quad (2.4)$$

Then V^0 is a toroidal vertex subalgebra of V ,

$$Y(v; x_0, \mathbf{x}) \in \mathcal{E}(V)[x_1^{\pm 1}, \dots, x_r^{\pm 1}] \quad \text{for } v \in V^0, \quad (2.5)$$

and $Y(\cdot; x_0, \mathbf{x})$ is injective on V^0 . Define a linear map $Y^0(\cdot, x_0) : V^0 \rightarrow \mathcal{E}(V)$ by

$$Y^0(v, x_0) = Y(v; x_0, \mathbf{x})|_{\mathbf{x}=\mathbf{1}} \quad \text{for } v \in V^0. \quad (2.6)$$

Then $(V^0, Y^0, \mathbf{1})$ carries the structure of a vertex algebra and (V, Y^0) carries the structure of a V^0 -module. Furthermore, for $v \in V$, $(m_0, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r$,

$$Y(v_{m_0, \mathbf{m}}\mathbf{1}; x_0, \mathbf{x}) = Y^0(v_{m_0, \mathbf{m}}\mathbf{1}, x_0)\mathbf{x}^{-\mathbf{m}} \in \mathbf{x}^{-\mathbf{m}}\mathcal{E}(V), \quad (2.7)$$

$$Y(v; x_0, \mathbf{x}) = \sum_{\mathbf{m} \in \mathbb{Z}^r} Y(v_{-1, \mathbf{m}}\mathbf{1}; x_0, \mathbf{x}). \quad (2.8)$$

We have the following analogue for V -modules:

Proposition 2.6. *Let V be a toroidal vertex algebra and let (W, Y_W) be a V -module. Then for $v \in V$, we have*

$$Y_W(v_{m_0, \mathbf{m}} \mathbf{1}; x_0, \mathbf{x}) \in \mathbf{x}^{-\mathbf{m}} \mathcal{E}(W) \quad \text{for } (m_0, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r \quad (2.9)$$

and

$$Y_W(v; x_0, \mathbf{x}) = \sum_{\mathbf{m} \in \mathbb{Z}^r} Y_W(v_{-1, \mathbf{m}} \mathbf{1}; x_0, \mathbf{x}). \quad (2.10)$$

For $u \in V^0$, set

$$Y_W^0(u, x_0) = Y_W(u; x_0, \mathbf{x})|_{\mathbf{x}=1} \in \mathcal{E}(W). \quad (2.11)$$

Then (W, Y_W^0) carries the structure of a module for V^0 viewed as a vertex algebra. Furthermore, if (W, Y_W) is irreducible, (W, Y_W^0) is irreducible.

Proof. For $v \in V$, with $Y_W(\mathbf{1}; x_0, \mathbf{x}) = 1_W$, from the Jacobi identity we have

$$\begin{aligned} & Y_W(Y(v; z_0, \mathbf{z}) \mathbf{1}; y_0, \mathbf{y}) \\ &= \text{Res}_{x_0} z_0^{-1} \delta \left(\frac{x_0 - y_0}{z_0} \right) Y_W(v; x_0, \mathbf{z}\mathbf{y}) - \text{Res}_{x_0} z_0^{-1} \delta \left(\frac{y_0 - x_0}{-z_0} \right) Y_W(v; x_0, \mathbf{z}\mathbf{y}) \\ &= \text{Res}_{x_0} x_0^{-1} \delta \left(\frac{y_0 + z_0}{x_0} \right) Y_W(v; x_0, \mathbf{z}\mathbf{y}) \\ &= Y_W(v; y_0 + z_0, \mathbf{z}\mathbf{y}) \\ &= e^{z_0 \frac{\partial}{\partial y_0}} Y_W(v; y_0, \mathbf{z}\mathbf{y}). \end{aligned} \quad (2.12)$$

Let $(m_0, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r$. We know $v_{m_0, \mathbf{m}} \mathbf{1} = 0$ if $m_0 \geq 0$. As for $m_0 < 0$, from (2.12) we get

$$\begin{aligned} Y_W(v_{-k-1, \mathbf{m}} \mathbf{1}; y_0, \mathbf{y}) &= \frac{1}{k!} \left(\frac{\partial}{\partial y_0} \right)^k \sum_{m_0 \in \mathbb{Z}} v_{m_0, \mathbf{m}} y_0^{-m_0-1} \mathbf{y}^{-\mathbf{m}} \\ &= \sum_{m_0 \in \mathbb{Z}} \binom{-m_0-1}{k} v_{m_0, \mathbf{m}} y_0^{-m_0-k-1} \mathbf{y}^{-\mathbf{m}} \end{aligned} \quad (2.13)$$

for $k \geq 0$. This proves the first part. Furthermore, using (2.13) we obtain

$$Y_W(v; y_0, \mathbf{y}) = \sum_{(m_0, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r} v_{m_0, \mathbf{m}} y_0^{-m_0-1} \mathbf{y}^{-\mathbf{m}} = \sum_{\mathbf{m} \in \mathbb{Z}^r} Y_W(v_{-1, \mathbf{m}} \mathbf{1}; y_0, \mathbf{y}),$$

which, though an infinite sum, exists in $\mathcal{E}(V, r)$ due to (2.9). The proof for the other assertions is straightforward. \square

The following basic facts immediately follow from Proposition 2.6 and its proof:

Lemma 2.7. *Let V be a toroidal vertex algebra. For $v \in V$, (m_0, \mathbf{m}) , $(n_0, \mathbf{n}) \in \mathbb{Z} \times \mathbb{Z}^r$, we have*

$$(v_{m_0, \mathbf{m}} \mathbf{1})_{n_0, \mathbf{n}} = 0 \quad \text{whenever } m_0 \geq 0 \text{ or } \mathbf{m} \neq \mathbf{n} \quad (2.14)$$

and

$$(v_{-k-1, \mathbf{m}} \mathbf{1})_{n_0, \mathbf{m}} = \binom{k - n_0 - 1}{k} v_{n_0 - k, \mathbf{m}} \quad (2.15)$$

on any V -module for $k \geq 0$. In particular, we have

$$(v_{-1, \mathbf{m}} \mathbf{1})_{-1, \mathbf{m}} \mathbf{1} = v_{-1, \mathbf{m}} \mathbf{1}. \quad (2.16)$$

As immediate consequences we have:

Corollary 2.8. *Let V be a toroidal vertex algebra. Then*

$$V^0 = \text{span}\{v_{-1, \mathbf{m}} \mathbf{1} \mid v \in V, \mathbf{m} \in \mathbb{Z}^r\}, \quad (2.17)$$

which also equals $\text{span}\{v_{-1, \mathbf{m}} \mathbf{1} \mid v \in V^0, \mathbf{m} \in \mathbb{Z}^r\}$.

Corollary 2.9. *Let V be a toroidal vertex algebra and let $v \in V$. Then $v \in V^0$ if and only if*

$$\begin{aligned} v_{m_0, \mathbf{m}} &= 0 \quad \text{for all } m_0 \in \mathbb{Z} \text{ and for all but finitely many } \mathbf{m} \in \mathbb{Z}^r, \\ v &= \sum_{\mathbf{m} \in \mathbb{Z}^r} v_{-1, \mathbf{m}} \mathbf{1}. \end{aligned}$$

For $v \in V$, $\mathbf{m} \in \mathbb{Z}^r$, we set

$$Y(v; x_0, \mathbf{m}) = \sum_{m_0 \in \mathbb{Z}} v_{m_0, \mathbf{m}} x_0^{-m_0-1}. \quad (2.18)$$

The following is a technical result which we shall need:

Lemma 2.10. *For $a, b \in V$, $m_0 \in \mathbb{Z}$, $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^r$, we have*

$$(a_{m_0, \mathbf{m}} b)_{-1, \mathbf{m} + \mathbf{n}} \mathbf{1} = a_{m_0, \mathbf{m}} b_{-1, \mathbf{n}} \mathbf{1}. \quad (2.19)$$

Proof. From the Jacobi identity for the pair (a, b) we get

$$\begin{aligned} & Y(Y(a; z_0, \mathbf{m})b; y_0, \mathbf{m} + \mathbf{n}) \mathbf{1} \\ &= \text{Res}_{x_0} z_0^{-1} \delta\left(\frac{x_0 - y_0}{z_0}\right) Y(a; x_0, \mathbf{m}) Y(b; y_0, \mathbf{n}) \mathbf{1} \\ &\quad - \text{Res}_{x_0} z_0^{-1} \delta\left(\frac{y_0 - x_0}{-z_0}\right) Y(b; y_0, \mathbf{n}) Y(a; x_0, \mathbf{m}) \mathbf{1} \\ &= \text{Res}_{x_0} z_0^{-1} \delta\left(\frac{x_0 - y_0}{z_0}\right) Y(a; x_0, \mathbf{m}) Y(b; y_0, \mathbf{n}) \mathbf{1} \\ &= Y(a; z_0 + y_0, \mathbf{m}) Y(b; y_0, \mathbf{n}) \mathbf{1}. \end{aligned}$$

Setting $y_0 = 0$ and then applying $\text{Res}_{z_0} z_0^{m_0}$ we obtain (2.19). \square

Recall that for a toroidal vertex algebra V , V^0 is a toroidal vertex algebra and a vertex algebra as well. Unless noted otherwise, a V^0 -module always stands for a module for V^0 viewed as a toroidal vertex algebra. Denote by $\text{Mod}V^0$ and Mod^0V^0 the module categories of V^0 viewed as a toroidal vertex algebra and as a vertex algebra, respectively.

The following is a connection between V -modules and V^0 -modules:

Proposition 2.11. *Let V be a toroidal vertex algebra. If (W, Y_W) is a V -module, then (W, Y'_W) is a V^0 -module where Y'_W is the restriction of Y_W , satisfying the condition that for every $v \in V$, $w \in W$, there exists an integer k such that*

$$x_0^k Y'_W(v_{-1, \mathbf{m}} \mathbf{1}; x_0, \mathbf{x}) w \in \mathbf{x}^{-\mathbf{m}} W[[x_0]] \quad \text{for all } \mathbf{m} \in \mathbb{Z}^r. \quad (2.20)$$

Conversely, if (W, Y'_W) is a V^0 -module satisfying the condition above, then (W, Y_W) is a V -module where

$$Y_W(v; x_0, \mathbf{x}) = \sum_{\mathbf{m} \in \mathbb{Z}^r} Y'_W(v_{-1, \mathbf{m}} \mathbf{1}; x_0, \mathbf{x}) \quad \text{for } v \in V. \quad (2.21)$$

Proof. Suppose that (W, Y_W) is a V -module. Then (W, Y'_W) is automatically a V^0 -module. For $v \in V$, $w \in W$, from Proposition 2.6, we have $Y_W(v_{-1, \mathbf{m}} \mathbf{1}; x_0, \mathbf{x}) w \in \mathbf{x}^{-\mathbf{m}} W((x_0))$ for $\mathbf{m} \in \mathbb{Z}^r$ and

$$Y_W(v; x_0, \mathbf{x}) w = \sum_{\mathbf{m} \in \mathbb{Z}^r} Y_W(v_{-1, \mathbf{m}} \mathbf{1}; x_0, \mathbf{x}) w.$$

Let k be an integer such that $x_0^k Y_W(v; x_0, \mathbf{x}) w \in W[[\mathbf{x}^{\pm 1}]][[x_0]]$. Then (2.20) follows.

Conversely, assume that (W, Y'_W) is a V^0 -module satisfying the very condition. Define a vertex operator map Y_W on V by using (2.21). Note that for $v \in V$, we have $Y_W(v; x_0, \mathbf{x}) \in \mathcal{E}(W, r)$ due to condition (2.20). It is clear that $Y_W(\mathbf{1}; x_0, \mathbf{x}) = 1_W$. Let $u, v \in V$, $m_0 \in \mathbb{Z}$, $\mathbf{m}, \mathbf{n}, \mathbf{p} \in \mathbb{Z}^r$. Using Lemma 2.7 and (2.19) we get

$$(u_{-1, \mathbf{m}} \mathbf{1})_{m_0, \mathbf{p}}(v_{-1, \mathbf{n}} \mathbf{1}) = \delta_{\mathbf{m}, \mathbf{p}} u_{m_0, \mathbf{m}} v_{-1, \mathbf{n}} \mathbf{1} = \delta_{\mathbf{m}, \mathbf{p}} (u_{m_0, \mathbf{m}} v)_{-1, \mathbf{m} + \mathbf{n}} \mathbf{1}. \quad (2.22)$$

That is,

$$Y(u_{-1, \mathbf{m}} \mathbf{1}; z_0, \mathbf{z})(v_{-1, \mathbf{n}} \mathbf{1}) = \text{Res}_{x_0} x_0^{-1} \mathbf{z}^{-\mathbf{m}} Y(Y(u; z_0, \mathbf{m})v; x_0, \mathbf{m} + \mathbf{n}) \mathbf{1}. \quad (2.23)$$

Using the Jacobi identity for Y'_W and this relation, we obtain

$$\begin{aligned}
& z_0^{-1} \delta \left(\frac{x_0 - y_0}{z_0} \right) Y_W(u; x_0, \mathbf{z}\mathbf{y}) Y_W(v; y_0, \mathbf{y}) \\
& - z_0^{-1} \delta \left(\frac{y_0 - x_0}{-z_0} \right) Y_W(v; y_0, \mathbf{y}) Y_W(u; x_0, \mathbf{z}\mathbf{y}) \\
= & z_0^{-1} \delta \left(\frac{x_0 - y_0}{z_0} \right) \sum_{\mathbf{m} \in \mathbb{Z}^r} \sum_{\mathbf{n} \in \mathbb{Z}^r} Y'_W(u_{-1, \mathbf{m}} \mathbf{1}; x_0, \mathbf{z}\mathbf{y}) Y'_W(v_{-1, \mathbf{n}} \mathbf{1}; y_0, \mathbf{y}) \\
& - z_0^{-1} \delta \left(\frac{y_0 - x_0}{-z_0} \right) \sum_{\mathbf{m} \in \mathbb{Z}^r} \sum_{\mathbf{n} \in \mathbb{Z}^r} Y'_W(v_{-1, \mathbf{n}} \mathbf{1}; y_0, \mathbf{y}) Y'_W(u_{-1, \mathbf{m}} \mathbf{1}; x_0, \mathbf{z}\mathbf{y}) \\
= & y_0^{-1} \delta \left(\frac{x_0 - z_0}{y_0} \right) \sum_{\mathbf{m} \in \mathbb{Z}^r} \sum_{\mathbf{n} \in \mathbb{Z}^r} Y'_W(Y(u_{-1, \mathbf{m}} \mathbf{1}; z_0, \mathbf{z})(v_{-1, \mathbf{n}} \mathbf{1}); y_0, \mathbf{y}) \\
= & y_0^{-1} \delta \left(\frac{x_0 - z_0}{y_0} \right) \sum_{\mathbf{m} \in \mathbb{Z}^r} \sum_{\mathbf{n} \in \mathbb{Z}^r} \text{Res}_{x_0} x_0^{-1} \mathbf{z}^{-\mathbf{m}} Y'_W(Y(Y(u; z_0, \mathbf{m})v; x_0, \mathbf{m} + \mathbf{n}) \mathbf{1}; y_0, \mathbf{y}) \\
= & y_0^{-1} \delta \left(\frac{x_0 - z_0}{y_0} \right) Y_W(Y(u; z_0, \mathbf{z})v; y_0, \mathbf{y}).
\end{aligned}$$

This proves that (W, Y_W) is a V -module. \square

As one of the main results of this section, we have:

Theorem 2.12. *Let V be a toroidal vertex algebra. If (W, Y_W) is a V^0 -module, then (W, Y_W^0) is a module for V^0 viewed as a vertex algebra, where*

$$Y_W^0(u, x_0) = Y_W(u; x_0, \mathbf{x})|_{\mathbf{x}=1} \quad \text{for } u \in V^0.$$

Conversely, if (W, Y_W^0) is a module for V^0 viewed as a vertex algebra, then (W, Y_W) is a V^0 -module where

$$Y_W(u; x_0, \mathbf{x}) = \sum_{\mathbf{m} \in \mathbb{Z}^r} Y_W^0(u_{-1, \mathbf{m}} \mathbf{1}, x_0) \mathbf{x}^{-\mathbf{m}} \quad \text{for } u \in V^0. \quad (2.24)$$

Furthermore, category $\text{Mod} V^0$ is naturally isomorphic to category $\text{Mod}^0 V^0$.

Proof. Let (W, Y_W) be a V^0 -module. By Proposition 2.6, for $u \in V^0$, we have

$$Y_W(u; x_0, \mathbf{x}) \in \mathcal{E}(W)[x_1^{\pm 1}, \dots, x_r^{\pm 1}],$$

so that $Y_W(u; x_0, \mathbf{x})|_{\mathbf{x}=1}$ exists in $\mathcal{E}(W)$. In view of this, $Y_W^0(\cdot, x_0)$ is a well defined linear map from V^0 to $\mathcal{E}(W)$. It can be readily seen that (W, Y_W^0) is a module for V^0 viewed as a vertex algebra. Let $\alpha : W_1 \rightarrow W_2$ be a homomorphism of V_0 -modules. For $v \in V^0$, $w \in W_1$, we have

$$\alpha(Y_{W_1}^0(v, x_0)w) = \alpha(Y_{W_1}(v; x_0, \mathbf{x})w|_{\mathbf{x}=1}) = Y_{W_2}(v; x_0, \mathbf{x})\alpha(w)|_{\mathbf{x}=1} = Y_{W_2}^0(v, x_0)\alpha(w).$$

This shows that α is also a homomorphism of modules for V^0 viewed as a vertex algebra. In this way, we get a functor \mathcal{A} from $\text{Mod} V^0$ to $\text{Mod}^0 V^0$.

On the other hand, let (W, Y_W^0) be a module for V^0 viewed as a vertex algebra. Define a linear map $Y_W(\cdot; x_0, \mathbf{x}) : V^0 \rightarrow \mathcal{E}(W, r)$ by

$$Y_W(u; x_0, \mathbf{x}) = \sum_{\mathbf{m} \in \mathbb{Z}^r} Y_W^0(u_{-1, \mathbf{m}} \mathbf{1}, x_0) \mathbf{x}^{-\mathbf{m}} \quad \text{for } u \in V^0. \quad (2.25)$$

Note that $u_{-1, \mathbf{m}} \mathbf{1} = 0$ for all but finitely many $\mathbf{m} \in \mathbb{Z}^r$ by Corollary 2.9, so that such defined $Y_W(u; x_0, \mathbf{x})$ indeed lies in $\mathcal{E}(W, r)$. Let $u, v \in V^0$. Using the Jacobi identity for Y_W^0 and (2.23), we obtain

$$\begin{aligned} & z_0^{-1} \delta \left(\frac{x_0 - y_0}{z_0} \right) Y_W(u; x_0, \mathbf{z}\mathbf{y}) Y_W(v; y_0, \mathbf{y}) \\ & - z_0^{-1} \delta \left(\frac{y_0 - x_0}{-z_0} \right) Y_W(v; y_0, \mathbf{y}) Y_W(u; x_0, \mathbf{z}\mathbf{y}) \\ = & z_0^{-1} \delta \left(\frac{x_0 - y_0}{z_0} \right) \sum_{\mathbf{m}, \mathbf{n} \in \mathbb{Z}^r} Y_W^0(u_{-1, \mathbf{m}} \mathbf{1}, x_0) Y_W^0(v_{-1, \mathbf{n}} \mathbf{1}, y_0) \mathbf{z}^{-\mathbf{m}} \mathbf{y}^{-\mathbf{m}-\mathbf{n}} \\ & - z_0^{-1} \delta \left(\frac{y_0 - x_0}{-z_0} \right) \sum_{\mathbf{m}, \mathbf{n} \in \mathbb{Z}^r} Y_W^0(v_{-1, \mathbf{n}} \mathbf{1}, y_0) Y_W^0(u_{-1, \mathbf{m}} \mathbf{1}, x_0) \mathbf{z}^{-\mathbf{m}} \mathbf{y}^{-\mathbf{m}-\mathbf{n}} \\ = & y_0^{-1} \delta \left(\frac{x_0 - z_0}{y_0} \right) \sum_{\mathbf{m}, \mathbf{n} \in \mathbb{Z}^r} Y_W^0(Y^0(u_{-1, \mathbf{m}} \mathbf{1}, z_0)(v_{-1, \mathbf{n}} \mathbf{1}), y_0) \mathbf{z}^{-\mathbf{m}} \mathbf{y}^{-\mathbf{m}-\mathbf{n}} \\ = & y_0^{-1} \delta \left(\frac{x_0 - z_0}{y_0} \right) \sum_{\mathbf{m}, \mathbf{n} \in \mathbb{Z}^r} Y_W^0(Y(u_{-1, \mathbf{m}} \mathbf{1}; z_0, \mathbf{z})(v_{-1, \mathbf{n}} \mathbf{1}), y_0) \mathbf{y}^{-\mathbf{m}-\mathbf{n}} \\ = & y_0^{-1} \delta \left(\frac{x_0 - z_0}{y_0} \right) \text{Res}_{x_0} x_0^{-1} \sum_{\mathbf{m}, \mathbf{n} \in \mathbb{Z}^r} Y_W^0(Y(Y(u; z_0, \mathbf{m})v; x_0, \mathbf{m} + \mathbf{n}) \mathbf{1}, y_0) \mathbf{z}^{-\mathbf{m}} \mathbf{y}^{-\mathbf{m}-\mathbf{n}} \\ = & y_0^{-1} \delta \left(\frac{x_0 - z_0}{y_0} \right) \sum_{\mathbf{m} \in \mathbb{Z}^r} Y_W(Y(u; z_0, \mathbf{m})v; y_0, \mathbf{y}) \mathbf{z}^{-\mathbf{m}} \\ = & y_0^{-1} \delta \left(\frac{x_0 - z_0}{y_0} \right) Y_W(Y(u; z_0, \mathbf{z})v; y_0, \mathbf{y}). \end{aligned}$$

This shows that that (W, Y_W) is a V^0 -module.

Let $\alpha : W \rightarrow W'$ be a homomorphism of modules for V^0 viewed as a vertex algebra. For any $v \in V$, $\mathbf{m} \in \mathbb{Z}^r$, from (2.25), using (2.9) and (2.10) we have

$$\begin{aligned} Y_W(v_{-1, \mathbf{m}} \mathbf{1}; x_0, \mathbf{x}) &= Y_W^0(v_{-1, \mathbf{m}} \mathbf{1}, x_0) \mathbf{x}^{-\mathbf{m}}, \\ Y_{W'}(v_{-1, \mathbf{m}} \mathbf{1}; x_0, \mathbf{x}) &= Y_{W'}^0(v_{-1, \mathbf{m}} \mathbf{1}, x_0) \mathbf{x}^{-\mathbf{m}}. \end{aligned}$$

It then follows that for any $v \in V$, $w \in W$, and $\mathbf{m} \in \mathbb{Z}^r$,

$$\alpha(Y_W(v_{-1, \mathbf{m}} \mathbf{1}; x_0, \mathbf{x})w) = Y_{W'}(v_{-1, \mathbf{m}} \mathbf{1}; x_0, \mathbf{x})\alpha(w).$$

This together with Corollary 2.8 implies that α is also a V^0 -module homomorphism. In this way, we get a functor \mathcal{B} from $\text{Mod}^0 V^0$ to $\text{Mod} V^0$. It is straightforward to see that \mathcal{A} is an isomorphism of categories with \mathcal{B} as its inverse. \square

As an immediate consequence of Proposition 2.5 we have:

Corollary 2.13. *Let V be a toroidal vertex algebra. Then $V^0 = V$ if and only if $Y(\cdot; x_0, \mathbf{x})$ on V is injective and*

$$Y(v; x_0, \mathbf{x})\mathbf{1} \in V[[x_0]][x_1^{\pm 1}, \dots, x_r^{\pm 1}] \quad \text{for all } v \in V. \quad (2.26)$$

Furthermore, using Corollary 2.13 and Proposition 2.5 we immediately have:

Corollary 2.14. *Let V be a toroidal vertex algebra. Then $(V^0)^0 = V^0$.*

The following is straightforward to prove:

Lemma 2.15. *Let V be a toroidal vertex algebra. Set*

$$K_Y(V) = \{v \in V \mid Y(v; x_0, \mathbf{x}) = 0\}. \quad (2.27)$$

Then

$$K_Y(V) = \{v \in V \mid Y(v; x_0, \mathbf{x})\mathbf{1} = 0\},$$

$K_Y(V)$ is an ideal of V , and $D(K_Y(V)) \subset K_Y(V)$ for any derivation D of V . On the other hand, for any V -module (W, Y_W) , we have

$$Y_W(v; x_0, \mathbf{x}) = 0 \quad \text{for } v \in K_Y(V).$$

Definition 2.16. For a toroidal vertex algebra V , we define

$$\overline{V} = V/K_Y(V), \quad (2.28)$$

a quotient toroidal vertex algebra.

Remark 2.17. Note that for any toroidal vertex algebra V , the quotient map from V to \overline{V} naturally gives rise to a functor η from the category of \overline{V} -modules to the category of V -modules. In view of Lemma 2.15, η is an isomorphism.

Lemma 2.18. *Suppose that V is a toroidal vertex algebra such that*

$$Y(v; x_0, \mathbf{x})\mathbf{1} \in V[[x_0]][x_1^{\pm 1}, \dots, x_r^{\pm 1}] \quad \text{for all } v \in V. \quad (2.29)$$

Then $\overline{V}^0 = \overline{V}$.

Proof. Let $v \in V$. From our assumption, $v_{-1, \mathbf{m}}\mathbf{1} = 0$ for all but finitely many $\mathbf{m} \in \mathbb{Z}^r$. On the other hand, by Proposition 2.5 we have

$$Y(v; x_0, \mathbf{x}) = \sum_{\mathbf{m} \in \mathbb{Z}^r} Y(v_{-1, \mathbf{m}}\mathbf{1}; x_0, \mathbf{x}) \quad \text{on } V.$$

Then $v - \sum_{\mathbf{m} \in \mathbb{Z}^r} v_{-1, \mathbf{m}}\mathbf{1} \in K_Y(V)$. It follows that $\overline{V} \subset \overline{V}^0$. Thus $\overline{V}^0 = \overline{V}$. \square

Next, we study the relation between the ideals of V and V^0 .

Lemma 2.19. *Let V be a toroidal vertex algebra. For any ideal I of V , set*

$$\mathcal{R}(I) = I \cap V^0. \quad (2.30)$$

Then $\mathcal{R}(I)$ is an ideal of V^0 and

$$\mathcal{R}(I) = \text{span}\{a_{m_0, \mathbf{m}} \mathbf{1} \mid a \in I, (m_0, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r\}. \quad (2.31)$$

On the other hand, for any ideal I^0 of V^0 , set

$$\mathcal{G}(I^0) = \text{span}\{a_{m_0, \mathbf{m}} v \mid a \in I^0, v \in V, (m_0, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r\}. \quad (2.32)$$

Then $\mathcal{G}(I^0)$ is an ideal of V and it is the ideal generated by I^0 .

Proof. It is clear that $\mathcal{R}(I)$ is an ideal of V^0 . Recall from Corollary 2.9 that for $a \in V^0$, $a = \sum_{\mathbf{m} \in \mathbb{Z}^r} a_{-1, \mathbf{m}} \mathbf{1}$, which is a finite sum. Using this we get

$$\mathcal{R}(I) = I \cap V^0 \subset \text{span}\{a_{m_0, \mathbf{m}} \mathbf{1} \mid a \in I, (m_0, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r\} \subset I \cap V^0 = \mathcal{R}(I),$$

proving (2.31).

Now, let I^0 be an ideal of V^0 . For $a \in I^0$ ($\subset V^0$), from Corollary 2.9, we have

$$a = \sum_{\mathbf{m} \in \mathbb{Z}^r} a_{-1, \mathbf{m}} \mathbf{1} \quad (\text{a finite sum}) \in \mathcal{G}(I^0).$$

This proves $I^0 \subset \mathcal{G}(I^0)$. Let $a \in I^0$, $u, v \in V$, $\mathbf{m} \in \mathbb{Z}^r$. Then there exists a nonnegative integer k such that

$$(x_0 - y_0)^k Y(u; x_0, \mathbf{m}) Y(a; y_0, \mathbf{y}) v = (x_0 - y_0)^k Y(a; y_0, \mathbf{y}) Y(u; x_0, \mathbf{m}) v. \quad (2.33)$$

Noticing that the right-hand side lies in $\mathcal{G}(I^0)[[x_0^{\pm 1}, y_i^{\pm 1} \mid 0 \leq i \leq r]]$, we get

$$(x_0 - y_0)^k Y(u; x_0, \mathbf{m}) Y(a; y_0, \mathbf{y}) v \in (\mathcal{G}(I^0)[[y_i^{\pm 1} \mid 1 \leq i \leq r]])((x_0))((y_0)).$$

Multiplying by $(x_0 - y_0)^{-k}$ we obtain

$$Y(u; x_0, \mathbf{m}) Y(a; y_0, \mathbf{y}) v \in (\mathcal{G}(I^0)[[y_i^{\pm 1} \mid 1 \leq i \leq r]])((x_0))((y_0)).$$

Then it follows that $\mathcal{G}(I^0)$ is a left ideal of V . Furthermore, for $a \in I^0$, $u, v \in V$, $\mathbf{m} \in \mathbb{Z}^r$, we have

$$\begin{aligned} & Y(Y(a; z_0, \mathbf{m}) u; y_0, \mathbf{y}) v \\ &= \text{Res}_{x_0 z_0}^{-1} \delta \left(\frac{x_0 - y_0}{z_0} \right) \mathbf{y}^{-\mathbf{m}} Y(a; x_0, \mathbf{m}) Y(u; y_0, \mathbf{y}) v \\ & \quad - \text{Res}_{x_0 z_0}^{-1} \delta \left(\frac{y_0 - x_0}{-z_0} \right) \mathbf{y}^{-\mathbf{m}} Y(u; y_0, \mathbf{y}) Y(a; x_0, \mathbf{m}) v. \end{aligned}$$

From this it follows that $\mathcal{G}(I^0)$ is also a right ideal of V . Therefore, $\mathcal{G}(I^0)$ is an ideal of V . It is clear that $\mathcal{G}(I^0)$ is the smallest ideal containing I^0 . \square

Furthermore, we have:

Proposition 2.20. *Let V be a toroidal vertex algebra. Then $\mathcal{R}(\mathcal{G}(I^0)) = I^0$ for any ideal I^0 of V^0 .*

Proof. Let I^0 be an ideal of V^0 . By Lemma 2.19, we have $I^0 \subset \mathcal{G}(I^0)$. Thus $I^0 \subset V^0 \cap \mathcal{G}(I^0) = \mathcal{R}(\mathcal{G}(I^0))$. On the other hand, let $u \in \mathcal{R}(\mathcal{G}(I^0)) = V^0 \cap \mathcal{G}(I^0)$. As $u \in V^0$, by Corollary 2.9 again we have

$$u = \sum_{\mathbf{m} \in \mathbb{Z}^r} u_{-1, \mathbf{m}} \mathbf{1} \quad (\text{a finite sum}). \quad (2.34)$$

As $u \in \mathcal{G}(I^0)$, by definition u is a linear combination of vectors of the form $a_{n_0, \mathbf{n}} v$ with $a \in I^0$, $v \in V$, $(n_0, \mathbf{n}) \in \mathbb{Z}^{r+1}$. Furthermore, we have

$$\begin{aligned} & Y(a_{n_0, \mathbf{n}} v; y_0, \mathbf{y}) \mathbf{1} \\ &= \text{Res}_{x_0} (x_0 - y_0)^{n_0} \mathbf{y}^{-\mathbf{n}} Y(a; x_0, \mathbf{n}) Y(v; y_0, \mathbf{y}) \mathbf{1} \\ &\quad - \text{Res}_{x_0} (-y_0 + x_0)^{n_0} \mathbf{y}^{-\mathbf{n}} Y(v; y_0, \mathbf{y}) Y(a; x_0, \mathbf{n}) \mathbf{1} \\ &= \text{Res}_{x_0} (x_0 - y_0)^{n_0} \mathbf{y}^{-\mathbf{n}} Y(a; x_0, \mathbf{n}) Y(v; y_0, \mathbf{y}) \mathbf{1}, \end{aligned}$$

which lies in $I^0[[y_0, y_1^{\pm 1}, \dots, y_r^{\pm 1}]]$ because $a \in I^0$, $Y(v; y_0, \mathbf{y}) \mathbf{1} \in V^0[[y_0, y_1^{\pm 1}, \dots, y_r^{\pm 1}]]$. It then follows from (2.34) that for every $u \in \mathcal{R}(\mathcal{G}(I^0))$, we have $u \in I^0$. Thus $\mathcal{R}(\mathcal{G}(I^0)) \subset I^0$, and hence $\mathcal{R}(\mathcal{G}(I^0)) = I^0$. \square

Note that from Proposition 2.20, \mathcal{G} gives rise to a one-to-one correspondence between the set of ideals of V^0 and the set of the equivalence classes of ideals of V where ideals I and J of V are said to be *equivalent* if $\mathcal{R}(I) = \mathcal{R}(J)$.

Let V be a toroidal vertex algebra. For an ideal I of V , set

$$K(I) = \{v \in V \mid v_{m_0, \mathbf{m}} V \subset I \text{ for all } (m_0, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r\}. \quad (2.35)$$

It is straightforward to show that $K(I)$ is an ideal containing I .

Proposition 2.21. *Let I be any ideal of V . Then*

$$K(\mathcal{G}(\mathcal{R}(I))) = K(I), \quad \mathcal{R}(I) = \mathcal{R}(K(I)).$$

Proof. From Lemma 2.19, we have

$$\mathcal{G}(\mathcal{R}(I)) = \text{span}\{a_{m_0, \mathbf{m}} v \mid a \in I \cap V^0, (m_0, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r, v \in V\} \subset I.$$

Set $K' = K(\mathcal{G}(\mathcal{R}(I)))$. As $\mathcal{G}(\mathcal{R}(I)) \subset I$, we have $K' \subset K(I)$. On the other hand, let $a \in K(I)$. For $(m_0, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r$, we have $a_{m_0, \mathbf{m}} \mathbf{1} \in I$ and $a_{m_0, \mathbf{m}} \mathbf{1} \in V^0$, so that

$$a_{m_0, \mathbf{m}} \mathbf{1} \in I \cap V^0 = \mathcal{R}(I) \subset \mathcal{G}(\mathcal{R}(I)).$$

It then follows from (2.8) that

$$Y(a; x_0, \mathbf{x})v = \sum_{\mathbf{m} \in \mathbb{Z}^r} Y(a_{-1, \mathbf{m}} \mathbf{1}; x_0, \mathbf{x})v \in \mathcal{G}(\mathcal{R}(I))[[x_0^{\pm 1}, \mathbf{x}^{\pm 1}]]$$

for all $v \in V$. Thus $a \in K'$. This proves $K(I) \subset K'$, and hence $K(I) = K'$.

As $I \subset K(I)$, we have $\mathcal{R}(I) \subset \mathcal{R}(K(I))$. Note that by Lemma 2.19 we have

$$\mathcal{R}(K(I)) = \{a_{m_0, \mathbf{m}} \mathbf{1} \mid a \in K(I), (m_0, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r\}.$$

Furthermore, for $a \in K(I)$, $(m_0, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r$, we have $a_{m_0, \mathbf{m}} \mathbf{1} \in I \cap V^0 = \mathcal{R}(I)$. Thus $\mathcal{R}(K(I)) \subset \mathcal{R}(I)$. Therefore $\mathcal{R}(I) = \mathcal{R}(K(I))$. \square

For an ideal of V , let π be the canonical map from V onto V/I . We see that $\pi^{-1}(K_Y(V/I)) = K(I)$. As an immediate consequence of Proposition 2.21 we have:

Corollary 2.22. *Let I and J be ideals of V . Then $\mathcal{R}(I) = \mathcal{R}(J)$ if and only if $K(I) = K(J)$, or equivalently, the identity map of V induces an isomorphism from $\overline{V/I}$ to $\overline{V/J}$.*

We have the following result on simplicity:

Proposition 2.23. *Let V be a toroidal vertex algebra. Then \overline{V} is a simple toroidal vertex algebra if and only if V^0 is a simple toroidal vertex algebra.*

Proof. Assume that V^0 is a simple toroidal vertex algebra. Let I be any ideal of \overline{V} . Then $I = J/K_Y(V)$ where J is an ideal of V , containing $K_Y(V)$. As V^0 is simple, we have either $J \cap V^0 = V^0$ or $J \cap V^0 = 0$. If $J \cap V^0 = V^0$, we have $\mathbf{1} \in V^0 \subset J$, which implies $J = V$. In this case, we have $I = \overline{V}$. Assume $J \cap V^0 = 0$. For any $u \in J$, we have $u_{m_0, \mathbf{m}} \mathbf{1} \in J \cap V^0 = 0$ for all $(m_0, \mathbf{m}) \in \mathbb{Z}^{r+1}$, that is, $u \in K_Y(V)$ by Lemma 2.15. Thus $J \subset K_Y(V)$. Consequently, $J = K_Y(V)$ and hence $I = 0$. Therefore, \overline{V} is simple.

Conversely, assume that \overline{V} is a simple toroidal vertex algebra. Let I^0 be any ideal of V^0 . As \overline{V} is simple, we have either $\mathcal{G}(I^0) + K_Y(V) = V$ or $\mathcal{G}(I^0) \subset K_Y(V)$. First, consider the case with $\mathcal{G}(I^0) + K_Y(V) = V$. Then there exist $u \in \mathcal{G}(I^0)$ and $v \in K_Y(V)$ such that $u + v = \mathbf{1}$. For any $w \in V$, we have

$$w = Y(\mathbf{1}; x_0, \mathbf{x})w = Y(u; x_0, \mathbf{x})w + Y(v; x_0, \mathbf{x})w = Y(u; x_0, \mathbf{x})w \in \mathcal{G}(I^0).$$

Thus $\mathcal{G}(I^0) = V$, which implies $I^0 = \mathcal{R}(\mathcal{G}(I^0)) = \mathcal{R}(V) = V^0$ by Proposition 2.20. We now consider the case with $\mathcal{G}(I^0) \subset K_Y(V)$. We have

$$I^0 = \mathcal{R}(\mathcal{G}(I^0)) \subset \mathcal{R}(K_Y(V)) = K_Y(V) \cap V^0 = 0.$$

Thus, $I^0 = 0$. This proves that V^0 is a simple toroidal vertex algebra. \square

Remark 2.24. For a toroidal vertex algebra V , denote by $\text{Mod}V$ the category of V -modules. The following are straightforward analogs of classical facts: Let $\varphi : V_1 \rightarrow V_2$ be a homomorphism of toroidal vertex algebras. Then any V_2 -module W is naturally a V_1 -module, which is denoted by $\text{Res}_\varphi(W)$. Moreover, if $\alpha : W \rightarrow W'$ is a homomorphism of V_2 -modules, then α is also a V_1 -module homomorphism from $\text{Res}_\varphi(W)$ to $\text{Res}_\varphi(W')$. Set $\text{Res}_\varphi(\alpha) = \alpha$. Then Res_φ is an exact faithful covariant functor from $\text{Mod}V_2$ to $\text{Mod}V_1$. Furthermore, if $\varphi : V_1 \rightarrow V_2$, $\psi : V_2 \rightarrow V_3$ are $(r+1)$ -toroidal vertex algebra homomorphisms, then $\text{Res}_\varphi \circ \text{Res}_\psi = \text{Res}_{\psi \circ \varphi}$.

Recall that a subcategory \mathcal{C}_1 of a category \mathcal{C} is called a *full subcategory* if $\text{Hom}_{\mathcal{C}}(A, B) = \text{Hom}_{\mathcal{C}_1}(A, B)$ for any objects A, B in \mathcal{C}_1 . Furthermore, a full subcategory \mathcal{C}_1 of \mathcal{C} is called a *strictly full subcategory* if whenever an object A in \mathcal{C} is isomorphic to an object of \mathcal{C}_1 , A belongs to \mathcal{C}_1 .

Here, we have:

Lemma 2.25. *Let V_1 and V_2 be toroidal vertex algebras and let $\varphi : V_1 \rightarrow V_2$ be a toroidal vertex algebra homomorphism such that $V_2^0 \subset \text{Im}\varphi$. Then Res_φ is an isomorphism from $\text{Mod}V_2$ to a strictly full subcategory of $\text{Mod}V_1$.*

Proof. Let W_1, W_2 be V_2 -modules. Assume that $\alpha : \text{Res}_\varphi(W_1) \rightarrow \text{Res}_\varphi(W_2)$ is a V_1 -module homomorphism, or equivalently, an $\text{Im}\varphi$ -module homomorphism. For $v \in V_2$, $w \in W_1$, $(m_0, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r$, using (2.15) we have

$$\alpha(v_{m_0, \mathbf{m}} w) = \alpha((v_{-1, \mathbf{m}} \mathbf{1})_{m_0, \mathbf{m}} w) = (v_{-1, \mathbf{m}} \mathbf{1})_{m_0, \mathbf{m}} \alpha(w) = v_{m_0, \mathbf{m}} \alpha(w)$$

since $v_{-1, \mathbf{m}} \mathbf{1} \in V_2^0 \subset \text{Im}\varphi$. This proves that α is also a V_2 -module homomorphism. Thus Res_φ is full. The proof for strict fullness is classical: Let W_1 be a V_1 -module, W_2 a V_2 -module, and $\alpha : \text{Res}_\varphi(W_2) \rightarrow W_1$ a V_1 -module isomorphism. For $u \in \ker \varphi$, $(m_0, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r$, we have

$$u_{m_0, \mathbf{m}} \alpha(w_2) = \alpha(u_{m_0, \mathbf{m}} w_2) = \alpha(\varphi(u)_{m_0, \mathbf{m}} w_2) = 0 \quad \text{for } w_2 \in W_2.$$

As $\alpha(W_2) = W_1$, we get $u_{m_0, \mathbf{m}} W_1 = 0$. Then W_1 is naturally an $\text{Im}\varphi$ -module. By using α , one can make W_1 a V_2 -module denoted by \tilde{W}_1 such that $\text{Res}_\varphi \tilde{W}_1 = W_1$. \square

Let V be a toroidal vertex algebra and let L be a quotient toroidal vertex algebra of V . Then we have the following commutative diagram

$$\begin{array}{ccc} V^0 & \longrightarrow & V \\ \downarrow & & \downarrow \\ L^0 & \longrightarrow & L \end{array}$$

As an immediate consequence of Proposition 2.11 and Lemma 2.25, we have:

Proposition 2.26. *A V -module W is naturally an L -module if and only if W is naturally an L^0 -module. Furthermore, $\text{Mod}L$ is the intersection of $\text{Mod}V$ and $\text{Mod}L^0$ as strictly full subcategories of $\text{Mod}V^0$.*

Recall from [LTW2] that a *vertex Leibniz algebra* is a vector space V equipped with a linear map $Y(\cdot, x) : V \rightarrow \mathcal{E}(V)$ such that the Jacobi identity for vertex algebras holds. Combining Remark 2.17, Corollary 2.18, and Theorem 2.12, we immediately have:

Corollary 2.27. *Suppose that V is a toroidal vertex algebra such that*

$$Y(v; x_0, \mathbf{x})\mathbf{1} \in V[[x_0]][x_1^{\pm 1}, \dots, x_r^{\pm 1}] \quad \text{for every } v \in V.$$

Then

$$Y(u; x_0, \mathbf{x})v \in V[[x_0^{\pm 1}]] [x_1^{\pm 1}, \dots, x_r^{\pm 1}] \quad \text{for any } u, v \in V.$$

For $u, v \in V$, define

$$Y^0(u, x_0)v = (Y(u; x_0, \mathbf{x})v)|_{\mathbf{x}=1}.$$

Then (V, Y^0) is a vertex Leibniz algebra and $\bar{V} = V/K_Y(V)$ is a vertex algebra. Furthermore, the category $\text{Mod}V$ is naturally isomorphic to the category of modules for \bar{V} viewed as a vertex algebra.

3 Irreducible modules for r -loop vertex algebras

In this section, we continue to study the structure of V^0 for a general $(r+1)$ -toroidal vertex algebra V . First, we show that V^0 has a canonical \mathbb{Z}^r -grading which makes V^0 a vertex \mathbb{Z}^r -graded algebra in a certain sense. Then we prove that the structure of an $(r+1)$ -toroidal vertex algebra V with $V^0 = V$ exactly amounts to that of a vertex \mathbb{Z}^r -graded algebra. We also study special vertex \mathbb{Z} -graded algebras which are r -loop vertex algebras and we classify their irreducible modules in terms of irreducible modules for the corresponding vertex algebras.

First, we study the $(r+1)$ -toroidal vertex algebra structure of V^0 for a general $(r+1)$ -toroidal vertex algebra V .

Definition 3.1. A *vertex \mathbb{Z}^r -graded algebra* is a vertex algebra V equipped with a \mathbb{Z}^r -grading $V = \bigoplus_{\mathbf{m} \in \mathbb{Z}^r} V_{(\mathbf{m})}$ such that $\mathbf{1} \in V_{(0)}$ and

$$u_k v \in V_{(\mathbf{m}+\mathbf{n})} \quad \text{for } u \in V_{(\mathbf{m})}, v \in V_{(\mathbf{n})}, \mathbf{m}, \mathbf{n} \in \mathbb{Z}^r, k \in \mathbb{Z}.$$

Proposition 3.2. *Let V be an $(r+1)$ -toroidal vertex algebra. Then $V^0 = \bigoplus_{\mathbf{m} \in \mathbb{Z}^r} V_{(\mathbf{m})}$ is a vertex \mathbb{Z}^r -graded algebra, where for $\mathbf{m} \in \mathbb{Z}^r$,*

$$V_{(\mathbf{m})} = \{v_{-1, -\mathbf{m}}\mathbf{1} \mid v \in V\}. \quad (3.1)$$

Proof. For $\mathbf{m} \in \mathbb{Z}^r$, define a linear map $\phi_{\mathbf{m}} : V \rightarrow V_{(-\mathbf{m})}$ by $\phi_{\mathbf{m}}(u) = u_{-1, \mathbf{m}} \mathbf{1}$ for $u \in V$. From Lemma 2.7, we have

$$\phi_{\mathbf{m}}|_{V_{(-\mathbf{n})}} = \delta_{\mathbf{m}, \mathbf{n}} \quad \text{for } \mathbf{m}, \mathbf{n} \in \mathbb{Z}^r.$$

It follows that $V^0 = \bigoplus_{\mathbf{m} \in \mathbb{Z}^r} V_{(\mathbf{m})}$ as a vector space. For $u, v \in V$, $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^r$, $m_0 \in \mathbb{Z}$, we have

$$\begin{aligned} \text{Res}_{x_0} x_0^{m_0} Y^0(u_{-1, \mathbf{m}} \mathbf{1}, x_0)(v_{-1, \mathbf{n}} \mathbf{1}) &= \text{Res}_{x_0} x_0^{m_0} Y(u_{-1, \mathbf{m}} \mathbf{1}; x_0, \mathbf{x})(v_{-1, \mathbf{n}} \mathbf{1})|_{\mathbf{x}=1} \\ &= \text{Res}_{x_0} x_0^{m_0} \mathbf{x}^{\mathbf{m}} Y(u_{-1, \mathbf{m}} \mathbf{1}; x_0, \mathbf{x})(v_{-1, \mathbf{n}} \mathbf{1}) \\ &= (u_{-1, \mathbf{m}} \mathbf{1})_{m_0, \mathbf{m}}(v_{-1, \mathbf{n}} \mathbf{1}), \end{aligned}$$

which by (2.22) is equal to $((u_{-1, \mathbf{m}} \mathbf{1})_{m_0, \mathbf{m}} v)_{-1, \mathbf{m} + \mathbf{n}} \mathbf{1} \in V_{(-\mathbf{m} - \mathbf{n})}$. This proves that V^0 is a vertex \mathbb{Z}^r -graded algebra. \square

On the other hand, we have the following result which is straightforward to prove:

Lemma 3.3. *Let $V = \bigoplus_{\mathbf{m} \in \mathbb{Z}^r} V_{(\mathbf{m})}$ be a vertex \mathbb{Z}^r -graded algebra. For $u \in V_{(\mathbf{m})}$ with $\mathbf{m} \in \mathbb{Z}^r$, define*

$$Y(u; x_0, \mathbf{x}) = Y(u, x_0) \mathbf{x}^{\mathbf{m}}.$$

Then V is an $(r+1)$ -toroidal vertex algebra with $V^0 = V$, which we denote by $F(V)$. Furthermore, if $\alpha : U \rightarrow V$ is a homomorphism of vertex \mathbb{Z}^r -graded algebras, then $\alpha : F(U) \rightarrow F(V)$ is also a homomorphism of $(r+1)$ -toroidal vertex algebras.

As an immediate consequence, we have:

Corollary 3.4. *Let V be a toroidal vertex algebra. Denote by D_0 the canonical derivation of V^0 viewed as a vertex algebra. Define linear operators D_1, \dots, D_r on V^0 by*

$$D_j(v) = m_j v \quad \text{for } v \in V_{(\mathbf{m})}, \quad \mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r. \quad (3.2)$$

Then V^0 equipped with derivations D_0, D_1, \dots, D_r is an extended toroidal vertex algebra, where D_1, \dots, D_r act on V^0 semi-simply with only integer eigenvalues. Furthermore, the extended toroidal vertex algebra structure on V^0 is unique.

Definition 3.5. Denote by \mathcal{C}_{r+1}^0 the subcategory of $(r+1)$ -toroidal vertex algebras, which consists of V satisfying the condition that $V^0 = V$.

In view of Lemma 3.3, we have a functor F from the category of \mathbb{Z}^r -graded vertex algebras to \mathcal{C}_{r+1}^0 . It is clear that F is an isomorphism. To summarize, we have:

Proposition 3.6. *The category \mathcal{C}_{r+1}^0 is a strictly full subcategory of the category of extended $(r+1)$ -toroidal vertex algebras and F is an isomorphism from the category of vertex \mathbb{Z}^r -graded algebras to \mathcal{C}_{r+1}^0 .*

From Corollary 2.7 and Proposition 2.16 in [LTW1] (including the skew-symmetry), we immediately have:

Corollary 3.7. *Let V be a toroidal vertex algebra and let I be a left ideal of V^0 . Then I is an ideal of V^0 (viewed as a toroidal vertex algebra) if and only if I is stable under the actions of D_0, \dots, D_r .*

Next, we give a canonical functor from the category of vertex algebras to the category of toroidal vertex algebras. Set

$$L_r = \mathbb{C}[t_1^{\pm 1}, \dots, t_r^{\pm 1}], \quad (3.3)$$

a commutative and associative algebra over \mathbb{C} . Note that any commutative and associative algebra A with identity 1 is naturally a vertex algebra with $Y(a, x)b = ab$ for $a, b \in A$ and with 1 as the vacuum. In particular, L_r is naturally a vertex algebra.

Remark 3.8. Let A be a commutative and associative algebra with identity 1. If W is an A -module, it can be readily seen that W is a module for A viewed as a vertex algebra. On the other hand, let (W, Y_W) be a module for A viewed as a vertex algebra. For any $a \in A$, we have (cf. [LL])

$$\frac{d}{dx} Y_W(a, x) = Y_W(Da, x) = 0,$$

as $Da = \left(\frac{d}{dx} Y(a, x)1\right)|_{x=0} = 0$. Then it is straightforward to show that W is a module for A viewed as an associative algebra. Therefore, a module for A viewed as an associative algebra is the same as a module for A viewed as a vertex algebra.

For any vector space W , we set

$$L_r(W) = W \otimes L_r.$$

Note that L_r is naturally an associative \mathbb{Z}^r -graded algebra and a vertex \mathbb{Z}^r -graded algebra. For any vertex algebra V , the tensor product vertex algebra $V \otimes L_r$ is naturally a vertex \mathbb{Z}^r -graded algebra. In view of Lemma 3.3, $L_r(V)$ is an extended $(r+1)$ -toroidal vertex algebra with

$$Y(v \otimes \mathbf{t}^{\mathbf{m}}; x_0, \mathbf{x}) = (Y(v, x_0) \otimes \mathbf{t}^{\mathbf{m}}) \mathbf{x}^{-\mathbf{m}} \quad (3.4)$$

and with derivations $D_0 = D \otimes 1$ and $D_i = -1 \otimes t_i \frac{\partial}{\partial t_i}$ for $1 \leq i \leq r$. Furthermore, for any vertex algebra homomorphism $\varphi : V_1 \rightarrow V_2$, $\varphi \otimes 1$ is an $(r+1)$ -toroidal vertex algebra homomorphism from $L_r(V_1)$ to $L_r(V_2)$. In this way, we have a covariant functor from the category of vertex algebras to the category of vertex \mathbb{Z}^r -graded algebras, and then to the category of $(r+1)$ -toroidal vertex algebras.

We have the following technical result:

Lemma 3.9. *Let V be a vertex \mathbb{Z}^r -graded algebra and let $\varphi : V \rightarrow A$ be a homomorphism of vertex algebras. Define a linear map $\tilde{\varphi} : V \rightarrow L_r(A)$ by*

$$\tilde{\varphi}(v) = \varphi(v) \otimes \mathbf{t}^{-\mathbf{m}} \quad \text{for } v \in V_{(\mathbf{m})} \text{ with } \mathbf{m} \in \mathbb{Z}^r.$$

Then $\tilde{\varphi}$ is a homomorphism of vertex \mathbb{Z}^r -graded algebras. Furthermore, if V is graded simple and if φ is not zero, then $\tilde{\varphi}$ is injective.

Proof. Let $u \in V_{(\mathbf{m})}$, $v \in V_{(\mathbf{n})}$, $k \in \mathbb{Z}$ with $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^r$. As V is a vertex \mathbb{Z}^r -graded algebra, we have $u_k v \in V_{(\mathbf{m}+\mathbf{n})}$. Then

$$\tilde{\varphi}(u_k v) = \varphi(u_k v) \otimes \mathbf{t}^{-\mathbf{m}-\mathbf{n}} = \varphi(u)_k \varphi(v) \otimes \mathbf{t}^{-\mathbf{m}-\mathbf{n}} = (\varphi(u) \otimes \mathbf{t}^{-\mathbf{m}})_k (\varphi(v) \otimes \mathbf{t}^{-\mathbf{n}}) = \tilde{\varphi}(u)_k \tilde{\varphi}(v).$$

Also, as $\mathbf{1} \in V_{(\mathbf{0})}$, we have $\tilde{\varphi}(\mathbf{1}) = \mathbf{1} \otimes 1$. Thus $\tilde{\varphi}$ is a homomorphism of vertex \mathbb{Z}^r -graded algebras. The last assertion is clear. \square

Remark 3.10. Let V be a vertex algebra. We have the tensor product vertex algebra $L_r(V)$. Note that an $L_r(V)$ -module structure on a vector space W amounts to a V -module structure $Y_W(\cdot, x)$ together with an L_r -module structure such that

$$aY_W(v, x)w = Y_W(v, x)(aw) \quad \text{for } a \in L_r, v \in V, w \in W.$$

We have the following straightforward analogue of the notion of evaluation module for affine Lie algebras:

Lemma 3.11. *Let V be a vertex algebra, let (W, Y_W) be a V -module, and let $\mathbf{a} \in (\mathbb{C}^\times)^r$. For $v \in V$, $\mathbf{m} \in \mathbb{Z}^r$, set*

$$\widehat{Y_W}(v \otimes \mathbf{t}^{\mathbf{m}}, x) = \mathbf{a}^{\mathbf{m}} Y_W(v, x). \quad (3.5)$$

Then $(W, \widehat{Y_W})$ is an $L_r(V)$ -module, which is denoted by $W_{\mathbf{a}}$. Furthermore, if W is an irreducible V -module, then $W_{\mathbf{a}}$ is an irreducible $L_r(V)$ -module.

Furthermore, we have:

Proposition 3.12. *Let V be a vertex algebra of countable dimension (over \mathbb{C}) and let W be an irreducible $L_r(V)$ -module. Then W as an $L_r(V)$ -module is isomorphic to $U_{\mathbf{a}}$ for some irreducible V -module U and for some $\mathbf{a} \in (\mathbb{C}^\times)^r$.*

Proof. Pick a nonzero vector w . Since W is an irreducible $L_r(V)$ -module, it follows (see [Li2], [DM]) that $W = \text{span}\{v_n w \mid v \in L_r(V), n \in \mathbb{Z}\}$. As $L_r(V)$ is of countable dimension, W is of countable dimension. Notice that L_r lies in the center of $L_r(V)$ viewed as a vertex algebra and that a module for L_r viewed as a vertex algebra is the same as a module for L_r viewed as an associative algebra. Then by the generalized Schur lemma, every element of L_r acts as a scalar on W , so that W is necessarily an irreducible V -module. Denote this irreducible V -module by U . Then we conclude that $W \simeq U_{\mathbf{a}}$ as an $L_r(V)$ -module for some $\mathbf{a} \in (\mathbb{C}^\times)^r$. \square

4 Toroidal vertex algebras associated to toroidal Lie algebras

In this section, we study toroidal vertex algebras associated to the r -loop algebras of affine Kac-Moody Lie algebras and we study their simple quotient toroidal vertex algebras.

Let \mathfrak{g} be a finite dimensional simple Lie algebra and let $\langle \cdot, \cdot \rangle$ be the killing form suitably normalized. Let $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t_0^{\pm 1}] \oplus \mathbb{C}\mathfrak{c}$ be the affine Lie algebra, where \mathfrak{c} is the standard central element and

$$[a \otimes t_0^m, b \otimes t_0^n] = [a, b] \otimes t_0^{m+n} + m\langle a, b \rangle \delta_{m+n,0} \mathfrak{c} \quad (4.1)$$

for $a, b \in \mathfrak{g}$, $m, n \in \mathbb{Z}$. Set

$$\tau = \hat{\mathfrak{g}} \otimes L_r = \hat{\mathfrak{g}} \otimes \mathbb{C}[t_1^{\pm 1}, \dots, t_r^{\pm 1}], \quad (4.2)$$

where

$$[u \otimes \mathbf{t}^{\mathbf{m}}, v \otimes \mathbf{t}^{\mathbf{n}}] = [u, v] \otimes \mathbf{t}^{\mathbf{m}+\mathbf{n}} \quad (4.3)$$

for $u, v \in \hat{\mathfrak{g}}$, $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^r$. Note that $L_r(\mathbb{C}\mathfrak{c}) (= \mathbb{C}\mathfrak{c} \otimes L_r)$ lies in the center of τ .

For $a \in \mathfrak{g}$, we set

$$a(x_0, \mathbf{x}) = \sum_{(m_0, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r} a(m_0, \mathbf{m}) x_0^{-m_0-1} \mathbf{x}^{-\mathbf{m}}, \quad (4.4)$$

where $a(m_0, \mathbf{m}) = a \otimes t_0^{m_0} \mathbf{t}^{\mathbf{m}}$. We also set

$$\mathfrak{c}(\mathbf{x}) = \sum_{\mathbf{m} \in \mathbb{Z}^r} \mathfrak{c}(\mathbf{m}) \mathbf{x}^{-\mathbf{m}}, \quad (4.5)$$

where $\mathfrak{c}(\mathbf{m}) = \mathfrak{c} \otimes \mathbf{t}^{\mathbf{m}}$. Then the Lie bracket relations (4.3) amount to

$$\begin{aligned} [a(x_0, \mathbf{x}), b(y_0, \mathbf{y})] &= [a, b](y_0, \mathbf{y}) x_0^{-1} \delta\left(\frac{y_0}{x_0}\right) \delta\left(\frac{\mathbf{y}}{\mathbf{x}}\right) \\ &\quad + \langle a, b \rangle \mathfrak{c}(\mathbf{y}) \frac{\partial}{\partial y_0} x_0^{-1} \delta\left(\frac{y_0}{x_0}\right) \delta\left(\frac{\mathbf{y}}{\mathbf{x}}\right), \end{aligned} \quad (4.6)$$

where $\delta\left(\frac{\mathbf{y}}{\mathbf{x}}\right) = \delta\left(\frac{y_1}{x_1}\right) \dots \delta\left(\frac{y_r}{x_r}\right)$.

As we need, we briefly recall from [LTW1] a conceptual construction. Let W be a vector space. An ordered pair $(a(x_0, \mathbf{x}), b(x_0, \mathbf{x}))$ in $\mathcal{E}(W, r)$ is said to be *compatible* if there exists non-negative integer k such that

$$(x_0 - y_0)^k a(x_0, \mathbf{x}) b(y_0, \mathbf{y}) \in \text{Hom}(W, W[[x_1^{\pm 1}, \dots, x_r^{\pm 1}, y_1^{\pm 1}, \dots, y_r^{\pm 1}]])((x_0, y_0)) \quad (4.7)$$

A subset U of $\mathcal{E}(W, r)$ is said to be *local* if for any $a(x_0, \mathbf{x}), b(x_0, \mathbf{x}) \in U$, there exists non-negative integer k such that

$$(x_0 - y_0)^k a(x_0, \mathbf{x}) b(y_0, \mathbf{y}) = (x_0 - y_0)^k b(y_0, \mathbf{y}) a(x_0, \mathbf{x}). \quad (4.8)$$

Notice that this commutation relation implies (4.7).

Definition 4.1. Let $a(x_0, \mathbf{x}), b(x_0, \mathbf{x}) \in \mathcal{E}(W, r)$. Assume $(a(x_0, \mathbf{x}), b(x_0, \mathbf{x}))$ is compatible. Define

$$a(x_0, \mathbf{x})_{m_0, \mathbf{m}} b(x_0, \mathbf{x}) \in \mathcal{E}(W, r) \quad \text{for } (m_0, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r$$

in terms of generating function

$$Y_{\mathcal{E}}(a(y_0, \mathbf{y}); z_0, \mathbf{z}) b(y_0, \mathbf{y}) = \sum_{(m_0, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r} a(y_0, \mathbf{y})_{m_0, \mathbf{m}} b(y_0, \mathbf{y}) z_0^{-m_0-1} \mathbf{z}^{-\mathbf{m}} \quad (4.9)$$

by

$$Y_{\mathcal{E}}(a(y_0, \mathbf{y}); z_0, \mathbf{z}) b(y_0, \mathbf{y}) = z_0^{-k} \left((x_0 - y_0)^k a(x_0, \mathbf{z}\mathbf{y}) b(y_0, \mathbf{y}) \right) \Big|_{x_0=y_0+z_0}, \quad (4.10)$$

where k is any non-negative integer such that (4.7) holds.

A τ -module W is called a *restricted module* if

$$a(x_0, \mathbf{x}) \in \mathcal{E}(W, r) \quad \text{for all } a \in \mathfrak{g}. \quad (4.11)$$

We have:

Lemma 4.2. *Let W be a restricted τ -module. Set*

$$U_W = \text{span}\{1_W, \mathfrak{c}(\mathbf{x}), a(x_0, \mathbf{x}) \mid a \in \mathfrak{g}\}. \quad (4.12)$$

Then U_W is a local subspace of $\mathcal{E}(W, r)$ and generates an $(r+1)$ -toroidal vertex algebra $\langle U_W \rangle$ with W as a faithful module. Moreover, $\langle U_W \rangle$ is a τ -module with $a(y_0, \mathbf{y})$ acting as $Y_{\mathcal{E}}(a(x_0, \mathbf{x}); y_0, \mathbf{y})$ for $a \in \mathfrak{g}$ and with $\mathfrak{c}(\mathbf{y})$ acting as $Y_{\mathcal{E}}(\mathfrak{c}(\mathbf{x}); y_0, \mathbf{y})$.

Proof. It follows from (4.6) that U_W is a local subspace of $\mathcal{E}(W, r)$. From [LTW1] (Theorem 3.4), U_W generates an $(r+1)$ -toroidal vertex algebra $(\langle U_W \rangle, Y_{\mathcal{E}}, 1_W)$ and W is a faithful $\langle U_W \rangle$ -module. The last assertion follows from the commutator relation (4.6) and the Proposition 3.9 of [LTW1]. \square

We shall need various subalgebras of τ . Set

$$\mathfrak{b} = \mathfrak{g} \otimes \mathbb{C}[t_0], \quad \mathfrak{n}_+ = \mathfrak{g} \otimes t_0 \mathbb{C}[t_0], \quad \mathfrak{n}_- = \mathfrak{g} \otimes t_0^{-1} \mathbb{C}[t_0^{-1}]. \quad (4.13)$$

Especially, we set

$$S_{\mathfrak{c}} = S(L_r(\mathbb{C}\mathfrak{c})) = S\left(\bigoplus_{\mathbf{m} \in \mathbb{Z}^r} \mathbb{C}(\mathfrak{c} \otimes \mathbf{t}^{\mathbf{m}})\right) \quad (4.14)$$

(the symmetric algebra).

We next introduce a special $L_r(\mathfrak{b})$ -module.

Lemma 4.3. *Set*

$$T = \mathfrak{g} \oplus \mathbb{C}\mathbf{c} \oplus \mathbb{C},$$

a vector space. Then T is an $L_r(\mathfrak{b})$ -module with

$$(a \otimes t_0^k \mathbf{t}^{\mathbf{m}}) \cdot b = \begin{cases} [a, b] & \text{if } k = 0 \\ \langle a, b \rangle \mathbf{c} & \text{if } k = 1 \\ 0 & \text{if } k \geq 2, \end{cases} \quad (4.15)$$

$$(a \otimes t_0^k \mathbf{t}^{\mathbf{m}}) \cdot (\mathbb{C}\mathbf{c} + \mathbb{C}) = 0$$

for $a, b \in \mathfrak{g}$, $k \geq 0$ and $\mathbf{m} \in \mathbb{Z}^r$.

Proof. It is straightforward to show (cf. [LTW1]) that $T (= \mathfrak{g} + \mathbb{C}\mathbf{c} + \mathbb{C})$ is a module for $\mathfrak{b} (= \mathfrak{g} \otimes \mathbb{C}[t_0])$ with

$$(a \otimes t_0^k) \cdot (b + \alpha \mathbf{c} + \beta) = \begin{cases} [a, b] & \text{if } k = 0 \\ \langle a, b \rangle \mathbf{c} & \text{if } k = 1 \\ 0 & \text{if } k \geq 2 \end{cases}$$

for $a, b \in \mathfrak{g}$, $\alpha, \beta \in \mathbb{C}$, $k \geq 0$. Then T becomes an $L_r(\mathfrak{b})$ -module through evaluation $t_i = 1$ for $i = 1, \dots, r$. That is,

$$(a \otimes t_0^k \mathbf{t}^{\mathbf{m}}) \cdot (b + \alpha \mathbf{c} + \beta) = \begin{cases} [a, b] & \text{if } k = 0 \\ \langle a, b \rangle \mathbf{c} & \text{if } k = 1 \\ 0 & \text{if } k \geq 2 \end{cases} \quad (4.16)$$

for $a, b \in \mathfrak{g}$, $\alpha, \beta \in \mathbb{C}$, $k \geq 0$ and $\mathbf{m} \in \mathbb{Z}^r$, as desired. \square

From the definition of T , $\mathfrak{g} + \mathbb{C}\mathbf{c}$ is an $L_r(\mathfrak{b})$ -submodule, and we have

$$T = (\mathfrak{g} + \mathbb{C}\mathbf{c}) \oplus \mathbb{C}, \quad (4.17)$$

as an $L_r(\mathfrak{b})$ -module. We shall always view $\mathfrak{g} + \mathbb{C}\mathbf{c}$ and \mathbb{C} as $L_r(\mathfrak{b})$ -submodules of T .

Form an induced τ -module

$$V(T, 0) = U(\tau) \otimes_{U(L_r(\mathfrak{b}))} T. \quad (4.18)$$

View T as a subspace of $V(T, 0)$, so that $\mathfrak{g} + \mathbb{C}\mathbf{c}$ and \mathbb{C} are subspaces of $V(T, 0)$. We see that

$$V(T, 0) = U(\tau)T = U(\tau)(\mathfrak{g} + \mathbb{C}\mathbf{c} + \mathbb{C}).$$

Set

$$\mathbf{1} = 1 \otimes 1 \in V(T, 0).$$

As the first main result of this section, we have:

Theorem 4.4. *There exists an $(r+1)$ -toroidal vertex algebra structure on $V(T, 0)$, which is uniquely determined by the condition that $\mathbf{1}$ is the vacuum vector and*

$$Y(\mathbf{c}; x_0, \mathbf{x}) = \mathbf{c}(\mathbf{x}), \quad Y(a; x_0, \mathbf{x}) = a(x_0, \mathbf{x}) \quad \text{for } a \in \mathfrak{g}.$$

Proof. The uniqueness is clear as $V(T, 0)$ as a τ -module is generated by T . Now, we establish the existence. Let W be any restricted τ -module. Recall from Lemma 4.2 the local subspace U_W of $\mathcal{E}(W, r)$ and the $(r+1)$ -toroidal vertex algebra $\langle U_W \rangle$ which is naturally a restricted τ -module. We next show that U_W is an $L_r(\mathfrak{b})$ -submodule of $\langle U_W \rangle$. For $a, b \in \mathfrak{g}$, $j \geq 0$, $\mathbf{m} \in \mathbb{Z}^r$, from the Proposition 3.9 of [LTW1] we have

$$\begin{aligned} a(x_0, \mathbf{x})_{j, \mathbf{m}} b(x_0, \mathbf{x}) &= \begin{cases} [a, b](x_0, \mathbf{x}) & \text{if } j = 0, \\ \langle a, b \rangle \mathbf{c}(\mathbf{x}) & \text{if } j = 1, \\ 0 & \text{if } j \geq 2, \end{cases} \\ a(x_0, \mathbf{x})_{j, \mathbf{m}} 1_W &= 0, \\ a(x_0, \mathbf{x})_{j, \mathbf{m}} \mathbf{c}(\mathbf{x}) &= 0. \end{aligned}$$

It follows that there exists an $L_r(\mathfrak{b})$ -module homomorphism $\phi : T \rightarrow U_W$ such that

$$\phi(a) = a(x_0, \mathbf{x}), \quad \phi(\mathbf{c}) = \mathbf{c}(\mathbf{x}), \quad \phi(1) = 1_W.$$

Then ϕ induces a τ -module homomorphism from $V(T, 0)$ into $\langle U_W \rangle$, which is denoted by $\phi_{x_0, \mathbf{x}}^W$. For $v \in V(T, 0)$, define

$$Y(v; x_0, \mathbf{x}) = \phi_{x_0, \mathbf{x}}^{V(T, 0)}(v). \quad (4.19)$$

Take $W = V(T, 0)$ and set $\phi_{x_0, \mathbf{x}} = \phi_{x_0, \mathbf{x}}^{V(T, 0)}$. Then for $a \in \mathfrak{g}$ and $v \in V(T, 0)$, we have

$$\begin{aligned} & Y(Y(a; z_0, \mathbf{z})v; y_0, \mathbf{y}) \\ &= \phi_{y_0, \mathbf{y}}(a(z_0, \mathbf{z})v) \\ &= Y_{\mathcal{E}}(a; y_0, \mathbf{y})\phi_{y_0, \mathbf{y}}(v) \\ &= \text{Res}_{x_0} z_0^{-1} \delta\left(\frac{x_0 - y_0}{z_0}\right) a(x_0, \mathbf{zy})\phi_{y_0, \mathbf{y}}(v) - z_0^{-1} \delta\left(\frac{y_0 - x_0}{-z_0}\right) \phi_{y_0, \mathbf{y}}(v) a(x_0, \mathbf{zy}) \\ &= \text{Res}_{x_0} z_0^{-1} \delta\left(\frac{x_0 - y_0}{z_0}\right) Y(a; x_0, \mathbf{zy}) Y(v; y_0, \mathbf{y}) \\ &\quad - \text{Res}_{x_0} z_0^{-1} \delta\left(\frac{y_0 - x_0}{-z_0}\right) Y(v; y_0, \mathbf{y}) Y(a; x_0, \mathbf{zy}). \end{aligned}$$

Now it follows immediately from [LTW1] (Theorem 3.10) that $(V(T, 0), Y, \mathbf{1})$ carries the structure of an $(r+1)$ -toroidal vertex algebra, as desired. \square

Furthermore, we have:

Theorem 4.5. *Let W be a restricted τ -module. Then there exists a $V(T, 0)$ -module structure on W , which is uniquely determined by*

$$Y_W(\mathbf{c}; x_0, \mathbf{x}) = \mathbf{c}(\mathbf{x}), \quad Y_W(a; x_0, \mathbf{x}) = a(x_0, \mathbf{x}) \quad \text{for } a \in \mathfrak{g}.$$

On the other hand, let (W, Y_W) be a $V(T, 0)$ -module. Then W becomes a restricted τ -module with

$$\mathbf{c}(\mathbf{x}) = Y_W(\mathbf{c}; x_0, \mathbf{x}), \quad a(x_0, \mathbf{x}) = Y_W(a; x_0, \mathbf{x}) \quad \text{for } a \in \mathfrak{g}.$$

Furthermore, the category of restricted τ -modules is isomorphic to $\text{Mod}V(T, 0)$.

Proof. We only need to prove the first two assertions. For the first assertion, the uniqueness is clear since $V(T, 0)$ as an $(r + 1)$ -toroidal vertex algebra is generated by T . We now establish the existence. For $a \in \mathfrak{g}$, $v \in V(T, 0)$, we have

$$\begin{aligned} \phi_{x_0, \mathbf{x}}^W(Y(a; y_0, \mathbf{y})v) &= \phi_{x_0, \mathbf{x}}^W(a(y_0, \mathbf{y})v) \\ &= Y_{\mathcal{E}}(a(x_0, \mathbf{x}); y_0, \mathbf{y})\phi_{x_0, \mathbf{x}}^W(v) = Y_{\mathcal{E}}(\phi_{x_0, \mathbf{x}}^W(a); y_0, \mathbf{y})\phi_{x_0, \mathbf{x}}^W(v). \end{aligned}$$

By the Lemma 2.10 of [LTW1], $Y_W(\cdot; x_0, \mathbf{x})$ is an $(r + 1)$ -toroidal vertex algebra module homomorphism from $V(T, 0)$ to $\langle U_W \rangle$. Since W is a canonical $\langle U_W \rangle$ -module, W becomes a $V(T, 0)$ -module with

$$Y_W(v; x_0, \mathbf{x}) = \phi_{x_0, \mathbf{x}}^W(v) \quad \text{for } v \in V(T, 0).$$

For the second assertion, combining (4.15) and the commutator formula (2.5) in [LTW1], we get

$$\begin{aligned} &[Y_W(a; x_0, \mathbf{x}), Y_W(b; y_0, \mathbf{y})] \\ &= \left(Y_W([a, b]; y_0, \mathbf{y})x_0^{-1}\delta\left(\frac{y_0}{x_0}\right) + \langle a, b \rangle Y_W(\mathbf{c}; y_0, \mathbf{y})\frac{\partial}{\partial y_0}x_0^{-1}\delta\left(\frac{y_0}{x_0}\right) \right) \delta\left(\frac{\mathbf{y}}{\mathbf{x}}\right) \end{aligned}$$

for $a, b \in \mathfrak{g}$. It follows that W is a restricted τ -module with $a(x_0, \mathbf{x}) = Y_W(a; x_0, \mathbf{x})$ for $a \in \mathfrak{g}$ and $\mathbf{c}(\mathbf{x}) = Y_W(\mathbf{c}; x_0, \mathbf{x})$. \square

Recall that \mathbb{C} is an $L_r(\mathfrak{b})$ -submodule of T . Set

$$V(S_{\mathbf{c}}, 0) = U(\tau) \otimes_{U(L_r(\mathfrak{b}))} \mathbb{C}, \tag{4.20}$$

which is naturally a τ -submodule of $V(T, 0)$.

Lemma 4.6. *The τ -submodule $V(S_{\mathbf{c}}, 0)$ of $V(T, 0)$ is an $(r + 1)$ -toroidal vertex subalgebra and we have $V(S_{\mathbf{c}}, 0) = V(T, 0)^0$.*

Proof. Recall that $\mathbf{1} = 1 \otimes 1 \in T \subset V(T, 0)$. We have $V(S_{\mathbf{c}}, 0) = U(\tau)\mathbf{1}$. It follows that $V(S_{\mathbf{c}}, 0)$ is the $(r + 1)$ -toroidal vertex subalgebra generated by the subspace $\mathfrak{g} + \mathbb{C}\mathbf{c} + \mathbb{C}\mathbf{1}$ ($= T$) of $V(T, 0)$. On the one hand, as $V(S_{\mathbf{c}}, 0) = U(\tau)\mathbf{1}$, we have $V(S_{\mathbf{c}}, 0) \subset V(T, 0)^0$. On the other hand, since $V(T, 0) = U(\tau)(\mathfrak{g} + \mathbb{C}\mathbf{c} + \mathbb{C}\mathbf{1})$, it follows from the Jacobi identity of $V(T, 0)$ that $V(T, 0)^0 \subset U(\tau)\mathbf{1} = V(S_{\mathbf{c}}, 0)$. \square

Combining Lemma 4.6 and Proposition 3.6, we immediately have:

Corollary 4.7. *$V(S_{\mathfrak{c}}, 0)$ is an extended $(r + 1)$ -toroidal vertex algebra such that $V(S_{\mathfrak{c}}, 0)^0 = V(S_{\mathfrak{c}}, 0)$, where the derivations D_0, D_1, \dots, D_r are given by*

$$\begin{aligned} D_0(v) &= \left(\frac{\partial}{\partial x_0} Y(v; x_0, \mathbf{x}) \mathbf{1} \right) \Big|_{x_0=0, \mathbf{x}=1}, \\ D_j(v) &= \left(x_j \frac{\partial}{\partial x_j} Y(v; x_0, \mathbf{x}) \mathbf{1} \right) \Big|_{x_0=0, \mathbf{x}=1} \end{aligned}$$

for $v \in V(S_{\mathfrak{c}}, 0)$, $1 \leq j \leq r$. On the other hand, $(V(S_{\mathfrak{c}}, 0), \mathbf{1}, Y^0)$ is a vertex algebra with D_0 as its D -operator, where

$$Y^0(v, x_0) = Y(v; x_0, \mathbf{x}) \Big|_{\mathbf{x}=1} \quad \text{for } v \in V(S_{\mathfrak{c}}, 0).$$

Noticing that $Y(a; x_0, \mathbf{x}) = a(x_0, \mathbf{x})$ for $a \in \mathfrak{g}$ and $Y(\mathfrak{c}; x_0, \mathbf{x}) = \mathfrak{c}(\mathbf{x})$ on $V(S_{\mathfrak{c}}, 0)$, we immediately have:

Corollary 4.8. *On $V(S_{\mathfrak{c}}, 0)$, the following relations hold for $a \in \mathfrak{g}$ and $1 \leq j \leq r$:*

$$\begin{aligned} [D_0, a(x_0, \mathbf{x})] &= \frac{\partial}{\partial x_0} a(x_0, \mathbf{x}) = -\frac{\partial}{\partial t_0} a(x_0, \mathbf{x}), \\ [D_j, a(x_0, \mathbf{x})] &= x_j \frac{\partial}{\partial x_j} a(x_0, \mathbf{x}) = -t_j \frac{\partial}{\partial t_j} a(x_0, \mathbf{x}), \\ [D_0, \mathfrak{c}(\mathbf{x})] &= \frac{\partial}{\partial x_0} \mathfrak{c}(\mathbf{x}) = 0, \\ [D_j, \mathfrak{c}(\mathbf{x})] &= x_j \frac{\partial}{\partial x_j} \mathfrak{c}(\mathbf{x}) = -t_j \frac{\partial}{\partial t_j} \mathfrak{c}(\mathbf{x}). \end{aligned}$$

We define a $\mathbb{Z} \times \mathbb{Z}^r$ -grading on τ by

$$\deg(a \otimes t_0^{m_0} \mathbf{t}^{\mathbf{m}}) = -(m_0, \mathbf{m}), \quad \deg(\mathfrak{c} \otimes \mathbf{t}^{\mathbf{m}}) = -(0, \mathbf{m}) \quad (4.21)$$

for $a \in \mathfrak{g}$, $(m_0, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r$. This makes τ a $\mathbb{Z} \times \mathbb{Z}^r$ -graded Lie algebra. Note that $L_r(\mathfrak{g} \otimes \mathbb{C}[t_0]) (= L_r(\mathfrak{b}))$ is a graded subalgebra. By assigning $\deg \mathbf{1} = (0, \mathbf{0})$, $V(S_{\mathfrak{c}}, 0)$ becomes a $\mathbb{Z} \times \mathbb{Z}^r$ -graded τ -module

$$V(S_{\mathfrak{c}}, 0) = \bigoplus_{(m_0, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r} V(S_{\mathfrak{c}}, 0)_{(m_0, \mathbf{m})}. \quad (4.22)$$

For $\mathbf{m} \in \mathbb{Z}^r$, set

$$V(S_{\mathfrak{c}}, 0)_{(\mathbf{m})} = \bigoplus_{m_0 \in \mathbb{Z}} V(S_{\mathfrak{c}}, 0)_{(m_0, \mathbf{m})}. \quad (4.23)$$

On the other hand, for $n \in \mathbb{Z}$, set

$$V(S_{\mathfrak{c}}, 0)_{(n)} = \bigoplus_{\mathbf{m} \in \mathbb{Z}^r} V(S_{\mathfrak{c}}, 0)_{(n, \mathbf{m})}. \quad (4.24)$$

We have:

Lemma 4.9. *Equipped with the \mathbb{Z} -grading $V(S_{\mathfrak{c}}, 0) = \bigoplus_{n \in \mathbb{Z}} V(S_{\mathfrak{c}}, 0)_{(n)}$, vertex algebra $V(S_{\mathfrak{c}}, 0)$ becomes a \mathbb{Z} -graded vertex algebra in the sense that $\mathbf{1} \in V(S_{\mathfrak{c}}, 0)_{(0)}$ and*

$$u_k v \in V(S_{\mathfrak{c}}, 0)_{(m+n-k-1)} \quad (4.25)$$

for $u \in V(S_{\mathfrak{c}}, 0)_{(m)}$, $v \in V(S_{\mathfrak{c}}, 0)_{(n)}$, $m, n, k \in \mathbb{Z}$. Furthermore, we have $V(S_{\mathfrak{c}}, 0)_{(n)} = 0$ for $n < 0$,

$$V(S_{\mathfrak{c}}, 0)_{(0)} = S_{\mathfrak{c}}, \text{ and } V(S_{\mathfrak{c}}, 0)_{(1)} = L_r(\mathfrak{g} \otimes t_0^{-1}) \otimes S_{\mathfrak{c}}. \quad (4.26)$$

Proof. It is clear that $\mathbf{1} \in V(S_{\mathfrak{c}}, 0)_{(0)}$. Note that for $a \in \mathfrak{g}$, $\mathbf{m} \in \mathbb{Z}^r$, we have

$$Y^0((a \otimes t_0^{-1} \mathbf{t}^{\mathbf{m}}) \mathbf{1}, x_0) = \left(\sum_{k \in \mathbb{Z}} (a \otimes t_0^k \mathbf{t}^{\mathbf{m}}) x_0^{-k-1} \mathbf{x}^{-\mathbf{m}} \right)_{\mathbf{x}=1} = \sum_{k \in \mathbb{Z}} (a \otimes t_0^k \mathbf{t}^{\mathbf{m}}) x_0^{-k-1},$$

$$Y^0((\mathfrak{c} \otimes \mathbf{t}^{\mathbf{m}}) \mathbf{1}, x_0) = \mathfrak{c} \otimes \mathbf{t}^{\mathbf{m}}.$$

It then follows that $V(S_{\mathfrak{c}}, 0)$ as a vertex algebra is generated by the subspace $L_r(\mathfrak{g} \otimes t_0^{-1}) \mathbf{1} + L_r(\mathfrak{c}) \mathbf{1}$. We see that (4.25) holds for $u \in L_r(\mathfrak{g} \otimes t_0^{-1}) \mathbf{1}$ and for $u \in L_r(\mathfrak{c}) \mathbf{1}$. Then it follows from induction that (4.25) holds for general u . This proves that $V(S_{\mathfrak{c}}, 0)$ is a \mathbb{Z} -graded vertex algebra. The furthermore assertion is clear. \square

For $0 \leq j \leq r$, denote by d_j the derivation $1 \otimes t_j \frac{\partial}{\partial t_j}$ on τ . Set

$$\tilde{\tau} = \tau \rtimes (\mathbb{C} d_0 + \cdots + \mathbb{C} d_r). \quad (4.27)$$

We have:

Lemma 4.10. *The $(r+1)$ -toroidal vertex algebra $V(S_{\mathfrak{c}}, 0)$ is a $\tilde{\tau}$ -module with d_1, \dots, d_r acting as $-D_1, \dots, -D_r$ and with d_0 acting as $-L(0)$, where $L(0)$ is the linear operator on $V(S_{\mathfrak{c}}, 0)$ defined by*

$$L(0)v = nv \quad \text{for } v \in V(S_{\mathfrak{c}}, 0)_{(n)}, \quad n \in \mathbb{Z}. \quad (4.28)$$

Proof. For $a \in \mathfrak{g}$, $(m_0, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r$, as

$$\deg(a \otimes t_0^{m_0} \mathbf{t}^{\mathbf{m}}) = -(m_0, \mathbf{m}), \quad \deg(\mathfrak{c} \otimes \mathbf{t}^{\mathbf{m}}) = -(0, \mathbf{m}),$$

we have

$$[L(0), (a \otimes t_0^{m_0} \mathbf{t}^{\mathbf{m}})] = -m_0(a \otimes t_0^{m_0} \mathbf{t}^{\mathbf{m}}), \quad [L(0), (\mathfrak{c} \otimes \mathbf{t}^{\mathbf{m}})] = 0$$

and $[L(0), D_j] = 0$ for $1 \leq j \leq r$, as needed. \square

Note that $V(T, 0)$ is a τ -module and an $S_{\mathfrak{c}}$ -module. For $u \in S_{\mathfrak{c}}$, denote by $\rho(u)$ the corresponding operator on $V(T, 0)$.

Lemma 4.11. *The subspace $S_{\mathfrak{c}}$ is a subalgebra of $V(S_{\mathfrak{c}}, 0)$ viewed as an $(r+1)$ -toroidal vertex algebra and a vertex algebra, where for $u \in (S_{\mathfrak{c}})_{\mathbf{m}}$ with $\mathbf{m} \in \mathbb{Z}^r$,*

$$Y(u; x_0, \mathbf{x}) = \rho(u)\mathbf{x}^{\mathbf{m}} \quad \text{and} \quad Y^0(u, x_0) = \rho(u). \quad (4.29)$$

Furthermore, every $V(S_{\mathfrak{c}}, 0)$ -module is naturally an $S_{\mathfrak{c}}$ -module.

Proof. Noticing that $\mathfrak{c} \in T \subset V(T, 0)$ and $Y(\mathfrak{c}; x_0, \mathbf{x}) = \mathfrak{c}(\mathbf{x})$, we have

$$\mathfrak{c}_{m_0, \mathbf{m}} = \delta_{m_0, -1} \rho(\mathfrak{c} \otimes \mathbf{t}^{\mathbf{m}}) \quad (4.30)$$

for $m_0 \in \mathbb{Z}$, $\mathbf{m} \in \mathbb{Z}^r$, which gives

$$\mathfrak{c}_{m_0, \mathbf{m}} \mathbf{1} = \delta_{m_0, -1} (\mathfrak{c} \otimes \mathbf{t}^{\mathbf{m}}) \in S_{\mathfrak{c}}.$$

Using this relation, formula (2.13), and (4.30) we get

$$Y(\mathfrak{c} \otimes \mathbf{t}^{\mathbf{m}}; x_0, \mathbf{x}) = Y(\mathfrak{c}_{-1, \mathbf{m}} \mathbf{1}; x_0, \mathbf{x}) = \rho(\mathfrak{c} \otimes \mathbf{t}^{\mathbf{m}}) \mathbf{x}^{-\mathbf{m}}. \quad (4.31)$$

Then we have

$$Y^0(\mathfrak{c} \otimes \mathbf{t}^{\mathbf{m}}, x_0) = Y(\mathfrak{c} \otimes \mathbf{t}^{\mathbf{m}}; x_0, \mathbf{x})|_{\mathbf{x}=1} = \rho(\mathfrak{c} \otimes \mathbf{t}^{\mathbf{m}}). \quad (4.32)$$

It then follows from induction and the Jacobi identity. \square

Next, we study simple quotient $(r+1)$ -toroidal vertex algebras of $V(T, 0)$. By Proposition 2.23, simple quotients of $V(T, 0)$ correspond to simple quotient $(r+1)$ -toroidal vertex algebras of $V(S_{\mathfrak{c}}, 0)$. Recall that every $V(S_{\mathfrak{c}}, 0)$ -module is naturally a module for $S_{\mathfrak{c}}$ as an associative algebra. On the other hand, to an $S_{\mathfrak{c}}$ -module we can associate a $V(S_{\mathfrak{c}}, 0)$ -module as follows: Let U be an $S_{\mathfrak{c}}$ -module. We let $L_r(\mathfrak{b})$ act on U trivially, making U an $L_r(\mathfrak{b} + \mathbb{C}\mathfrak{c})$ -module. Then set

$$M(U) = U(\tau) \otimes_{U(L_r(\mathfrak{b} + \mathbb{C}\mathfrak{c}))} U, \quad (4.33)$$

which is a restricted τ -module. In view of Theorem 4.5, $M(U)$ is naturally a $V(T, 0)$ -module and a $V(S_{\mathfrak{c}}, 0)$ -module. In particular, we have $V(S_{\mathfrak{c}}, 0) = M(S_{\mathfrak{c}})$.

Remark 4.12. Note that every irreducible τ -module is of countable dimension over \mathbb{C} . As $S_{\mathfrak{c}}$ lies in the center of $U(\tau)$, each element of $S_{\mathfrak{c}}$ necessarily acts as a scalar on every irreducible τ -module. Thus, an algebra homomorphism $\psi : S_{\mathfrak{c}} \rightarrow \mathbb{C}$ is attached to every irreducible τ -module.

Now, we establish some basic results.

Lemma 4.13. *An ideal of $V(S_{\mathfrak{c}}, 0)$ viewed as an $(r+1)$ -toroidal vertex algebra (resp, a vertex algebra) exactly amounts to a D_0 -stable and \mathbb{Z}^r -graded (resp. D_0 -stable) τ -submodule.*

Proof. As $V(S_{\mathfrak{c}}, 0)$ is generated from $\mathbf{1}$ by the coefficients of vertex operators

$$Y(a; x_0, \mathbf{x}) (= a(x_0, \mathbf{x})) \quad (a \in \mathfrak{g}) \quad \text{and} \quad Y(\mathfrak{c}; x_0, \mathbf{x}) (= \mathfrak{c}(\mathbf{x})),$$

it follows that a left ideal of $V(S_{\mathfrak{c}}, 0)$ amounts to a τ -submodule. Then by Corollary 3.7, an ideal of $V(S_{\mathfrak{c}}, 0)$ viewed as an $(r+1)$ -toroidal vertex algebra (resp. a vertex algebra) amounts to a D_0 -stable and \mathbb{Z}^r -graded (resp. D_0 -stable) τ -submodule. \square

Lemma 4.14. *Let K be a \mathbb{Z}^r -graded ideal (resp. an ideal) of $S_{\mathfrak{c}}$. Then $U(\tau)K$ is a \mathbb{Z} -graded ideal of $V(S_{\mathfrak{c}}, 0)$ viewed as an $(r+1)$ -toroidal vertex algebra (resp. a vertex algebra).*

Proof. Notice that $L(-1)S_{\mathfrak{c}} = 0$, which implies $L(-1)K = 0$. It follows that $L(-1)(U(\tau)K) \subset U(\tau)K$. Then $U(\tau)K$ is an ideal of $V(S_{\mathfrak{c}}, 0)$ viewed as a vertex algebra. If K is \mathbb{Z}^r -graded, then $U(\tau)K$ is also \mathbb{Z}^r -graded. By Lemma 4.13, $U(\tau)K$ is an ideal of $V(S_{\mathfrak{c}}, 0)$ viewed as an $(r+1)$ -toroidal vertex algebra in this case. \square

On the other hand, we have:

Lemma 4.15. *Let J be a maximal \mathbb{Z} -graded ideal of $V(S_{\mathfrak{c}}, 0)$ viewed as an $(r+1)$ -toroidal vertex algebra (resp. a vertex algebra). Then $J_{(0)}$ is a maximal \mathbb{Z}^r -graded ideal (resp. maximal ideal) of $S_{\mathfrak{c}}$.*

Proof. Let I be any proper \mathbb{Z}^r -graded ideal of $S_{\mathfrak{c}}$, containing $J_{(0)}$. By Lemma 4.14, $U(\tau)I$ is a $\mathbb{Z} \times \mathbb{Z}^r$ -graded ideal of $V(S_{\mathfrak{c}}, 0)$ with $(U(\tau)I)_{(0)} = I$. Then $J + U(\tau)I$ is a \mathbb{Z}^r -graded ideal of $V(S_{\mathfrak{c}}, 0)$, containing J . Since $(J + U(\tau)I)_{(0)} = I$, $J + U(\tau)I$ is a proper \mathbb{Z} -graded ideal. It follows that $J + U(\tau)I = J$. Thus $I = J_{(0)}$. This proves that $J_{(0)}$ is maximal. \square

Definition 4.16. Let $A = S_{\mathfrak{c}}/I$ be a quotient algebra of $S_{\mathfrak{c}}$ (with I an ideal of $S_{\mathfrak{c}}$). Define

$$V(A, 0) = V(S_{\mathfrak{c}}, 0)/U(\tau)I \simeq U(\tau) \otimes_{U(L_r(\mathfrak{b} + \mathfrak{c}_{\mathfrak{c}}))} A, \quad (4.34)$$

which is a vertex algebra by Lemma 4.14. If A is also \mathbb{Z}^r -graded, $V(A, 0)$ is an $(r+1)$ -toroidal vertex algebra.

Proposition 4.17. *For any \mathbb{Z}^r -graded simple quotient algebra A of $S_{\mathfrak{c}}$, $V(A, 0)$ has a unique maximal $\mathbb{Z} \times \mathbb{Z}^r$ -graded ideal $J(A)$. Denote by $L(A, 0)$ the \mathbb{Z} -graded simple quotient $(r+1)$ -toroidal vertex algebra. Furthermore, any simple \mathbb{Z} -graded quotient of $(r+1)$ -toroidal vertex algebra $V(S_{\mathfrak{c}}, 0)$ is isomorphic to $L(A, 0)$ for some \mathbb{Z}^r -graded simple quotient algebra A of $S_{\mathfrak{c}}$.*

Proof. Note that $V(A, 0)$ is naturally $\mathbb{Z} \times \mathbb{Z}^r$ -graded with $A = V(A, 0)_{(0)}$. Let $J(A)$ be the sum of all $\mathbb{Z} \times \mathbb{Z}^r$ -graded ideals P of $V(A, 0)$ such that $P_{(0)} = 0$. It is straightforward to see that $J(A)$ is the unique maximal $\mathbb{Z} \times \mathbb{Z}^r$ -graded ideal. Now,

let J be any maximal \mathbb{Z} -graded ideal of $(r+1)$ -toroidal vertex algebra $V(S_{\mathfrak{c}}, 0)$. By Lemma 4.13, J is $\mathbb{Z} \times \mathbb{Z}^r$ -graded. Set $J_{(0)} = \bigoplus_{\mathbf{m} \in \mathbb{Z}^r} J_{(0, \mathbf{m})}$. By Lemma 4.15, $J_{(0)}$ is a maximal \mathbb{Z}^r -graded ideal of $S_{\mathfrak{c}}$. We have $(V(S_{\mathfrak{c}}, 0)/J)_{(0)} = S_{\mathfrak{c}}/J_{(0)}$,

$$L_r(\mathfrak{b})(S_{\mathfrak{c}}/J_{(0)}) = 0, \quad \text{and} \quad V(S_{\mathfrak{c}}, 0)/J = U(\tau)(S_{\mathfrak{c}}/J_{(0)}).$$

Set $A = S_{\mathfrak{c}}/J_{(0)}$. By the construction of $V(A, 0)$, there exists a τ -module homomorphism from $V(A, 0)$ to $V(S_{\mathfrak{c}}, 0)/J$, extending the identity map on A . It follows that $V(S_{\mathfrak{c}}, 0)/J \simeq L(A, 0)$. \square

Remark 4.18. In view of Proposition 4.17, to determine simple \mathbb{Z} -graded quotients of $(r+1)$ -toroidal vertex algebra $V(S_{\mathfrak{c}}, 0)$, it suffices to determine \mathbb{Z}^r -graded simple quotients of $S_{\mathfrak{c}}$. Note that every \mathbb{Z}^r -graded simple quotient algebra of $S_{\mathfrak{c}}$ is isomorphic to the image of a \mathbb{Z}^r -graded algebra homomorphism from $S_{\mathfrak{c}}$ to L_r and that a \mathbb{Z}^r -graded simple quotient of $S_{\mathfrak{c}}$ is an irreducible \mathbb{Z}^r -graded $S_{\mathfrak{c}}$ -module. It was proved by Rao ([R1], Lemma 3.3) that every irreducible \mathbb{Z}^r -graded $S_{\mathfrak{c}}$ -module is isomorphic to $\text{Im}\psi$ for some \mathbb{Z}^r -graded algebra homomorphism $\psi : S_{\mathfrak{c}} \rightarrow L_r$.

Definition 4.19. Let $\psi : S_{\mathfrak{c}} \rightarrow L_r$ be a \mathbb{Z}^r -graded algebra homomorphism. Define $V(\psi, 0)$ to be the quotient $(r+1)$ -toroidal vertex algebra of $V(S_{\mathfrak{c}}, 0)$ modulo the ideal generated by $\ker \psi$, which is $U(\tau)(\ker \psi)$.

Alternatively, we have

$$V(\psi, 0) = V(A_{\psi}, 0) = U(\tau) \otimes_{U(L_r(\mathfrak{b} + \mathbb{C}\mathfrak{c}))} A_{\psi}, \quad (4.35)$$

where $A_{\psi} = S_{\mathfrak{c}}/\text{Ker}\psi \simeq \text{Im}\psi$. As $U(\tau)(\ker \psi)$ is a $\mathbb{Z} \times \mathbb{Z}^r$ -graded ideal, $V(\psi, 0)$ is \mathbb{Z} -graded with

$$V(\psi, 0)_{(0)} = A_{\psi} \quad \text{and} \quad V(\psi, 0)_{(n)} = 0 \quad \text{for } n < 0. \quad (4.36)$$

In view of Rao's result, to determine simple \mathbb{Z} -graded quotients of $V(S_{\mathfrak{c}}, 0)$, we need to determine simple quotients of $V(\psi, 0)$ for all \mathbb{Z}^r -graded algebra homomorphisms $\psi : S_{\mathfrak{c}} \rightarrow L_r$ such that $\text{Im}\psi$ are \mathbb{Z}^r -graded simple $S_{\mathfrak{c}}$ -modules.

Lemma 4.20. *As a τ -module, $V(\psi, 0)$ has a unique maximal $\mathbb{Z} \times \mathbb{Z}^r$ -graded τ -submodule, which is denoted by M_{ψ} . Moreover, M_{ψ} is the unique maximal \mathbb{Z} -graded ideal of $(r+1)$ -toroidal vertex algebra $V(\psi, 0)$.*

Proof. Note that $V(\psi, 0)_{(0)} = A_{\psi} \simeq \text{Im}\psi$, which is a \mathbb{Z}^r -graded irreducible $S_{\mathfrak{c}}$ -module and that $V(\psi, 0) = U(\tau)A_{\psi}$. In view of this, for every proper $\mathbb{Z} \times \mathbb{Z}^r$ -graded τ -submodule W of $V(\psi, 0)$, we have $W_{(0)} = 0$. Then it follows that the sum M_{ψ} of all proper $\mathbb{Z} \times \mathbb{Z}^r$ -graded τ -submodules of $V(\psi, 0)$ is the (unique) maximal proper $\mathbb{Z} \times \mathbb{Z}^r$ -graded τ -submodule. To prove that M_{ψ} is an ideal of $V(\psi, 0)$, by Lemma 4.13, we need to show $D_0(M_{\psi}) \subset M_{\psi}$. Since $D_0V(\psi, 0)_{(n)} \subset V(\psi, 0)_{(n+1)}$ for $n \in \mathbb{Z}$ and $(M_{\psi})_{(0)} = 0$, we have

$$\mathbb{C}[D_0]M_{\psi} \subset \bigoplus_{k>0} V(\psi, 0)_{(k)}.$$

Note that

$$[D_0, a \otimes t_0^k \mathbf{t}^{\mathbf{m}}] = -k(a \otimes t_0^{k-1} \mathbf{t}^{\mathbf{m}}), \quad [D_0, \mathbf{c} \otimes \mathbf{t}^{\mathbf{m}}] = 0$$

for $a \in \mathfrak{g}$, $k \in \mathbb{Z}$, $\mathbf{m} \in \mathbb{Z}^r$. It follows that $\mathbb{C}[D_0]M_\psi$ is a proper $\mathbb{Z} \times \mathbb{Z}^r$ -graded τ -submodule. From the definition of M_ψ , we have $\mathbb{C}[D_0]M_\psi \subset M_\psi$, which implies $D_0 M_\psi \subset M_\psi$. By Lemma 4.13, M_ψ is an ideal. \square

Definition 4.21. Let $\psi : S_{\mathbf{c}} \rightarrow L_r$ be a \mathbb{Z}^r -graded algebra homomorphism such that $\text{Im} \psi$ is a \mathbb{Z}^r -graded simple $S_{\mathbf{c}}$ -module. Define $L(\psi, 0)$ to be the unique simple \mathbb{Z} -graded quotient $(r+1)$ -toroidal vertex algebra of $V(\psi, 0)$.

Let $E(1) : L_r \rightarrow \mathbb{C}$ be the evaluation map with $t_j = 1$ for $1 \leq j \leq r$. For any algebra homomorphism $\psi : S_{\mathbf{c}} \rightarrow L_r$, we set

$$\bar{\psi} = E(1) \circ \psi, \quad (4.37)$$

an algebra homomorphism from $S_{\mathbf{c}}$ to \mathbb{C} . Conversely, any algebra homomorphism arises this way from a \mathbb{Z}^r -graded algebra homomorphism ψ .

Definition 4.22. Let $\psi : S_{\mathbf{c}} \rightarrow L_r$ be an algebra homomorphism. We define

$$\bar{V}(\psi, 0) = V(S_{\mathbf{c}}, 0) / U(\tau)(\text{Ker} \bar{\psi}) = U(\tau) \otimes_{U(L_r(\mathfrak{b} + \mathbb{C}\mathbf{c}))} (S_{\mathbf{c}} / \text{ker } \bar{\psi}), \quad (4.38)$$

a quotient vertex algebra of $V(S_{\mathbf{c}}, 0)$.

Setting $A_{\bar{\psi}} = S_{\mathbf{c}} / (\text{ker } \bar{\psi})$, we have

$$\bar{V}(\psi, 0) = V(A_{\bar{\psi}}, 0) = U(\tau) \otimes_{U(L_r(\mathfrak{b} + \mathbb{C}\mathbf{c}))} A_{\bar{\psi}}. \quad (4.39)$$

Note that $U(\tau)(\text{ker } \bar{\psi})$ is a \mathbb{Z} -graded ideal. Thus $\bar{V}(\psi, 0)$ is \mathbb{Z} -graded with

$$\bar{V}(\psi, 0)_{(0)} = A_{\bar{\psi}} \quad \text{and} \quad \bar{V}(\psi, 0)_{(n)} = 0 \quad \text{for } n < 0.$$

As $\text{ker } \psi \subset \text{ker } \bar{\psi}$, $\bar{V}(\psi, 0)$ is naturally a homomorphism image of $V(\psi, 0)$.

Similarly, we have:

Lemma 4.23. *As a τ -module, $\bar{V}(\psi, 0)$ has a unique maximal \mathbb{Z} -graded τ -submodule denoted by $M_{\bar{\psi}}$, which is the unique maximal \mathbb{Z} -graded ideal of vertex algebra $\bar{V}(\psi, 0)$.*

Define

$$L^0(\psi, 0) = \bar{V}(\psi, 0) / M_{\bar{\psi}}, \quad (4.40)$$

a simple \mathbb{Z} -graded vertex algebra.

We have:

Corollary 4.24. *Let $\psi : S_{\mathbf{c}} \rightarrow L_r$ be a \mathbb{Z}^r -graded algebra homomorphism such that $\text{Im} \psi$ is a \mathbb{Z}^r -graded simple \mathbb{Z} -graded $S_{\mathbf{c}}$ -module. Then $L^0(\psi, 0)$ is isomorphic to a simple \mathbb{Z} -graded quotient vertex algebra of $L(\psi, 0)$.*

Recall that $L^0(\psi, 0)$ is the unique simple \mathbb{Z} -graded quotient vertex algebra of $\overline{V}(\psi, 0)$ and that $L^0(\psi, 0)$ is a \mathbb{Z} -graded irreducible τ -module generated by vector 1_ψ satisfying the condition:

$$L_r(\mathfrak{b})1_\psi = 0, \quad u \cdot 1_\psi = \bar{\psi}(u)1_\psi \quad \text{for } u \in S_\mathfrak{c}.$$

Let γ be a linear functional on L_r . We define

$$I_\gamma = \{u \in L_r \mid \gamma(uL_r) = 0\}. \quad (4.41)$$

It can be readily seen that I_γ is an ideal of L_r such that $\gamma(I_\gamma) = 0$ and it is the (unique) maximal ideal with this property. Define a bilinear form $\langle \cdot, \cdot \rangle_\gamma$ on $L_r(\mathfrak{g})$ by

$$\langle a \otimes \mathfrak{t}^{\mathbf{m}}, b \otimes \mathfrak{t}^{\mathbf{n}} \rangle_\gamma = \langle a, b \rangle \gamma(\mathfrak{t}^{\mathbf{m}+\mathbf{n}}) \quad (4.42)$$

for $a, b \in \mathfrak{g}$, $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^r$. It is clear that $\langle \cdot, \cdot \rangle_\gamma$ is a symmetric invariant bilinear form on $L_r(\mathfrak{g})$. Furthermore, we have:

Lemma 4.25. *The kernel of the bilinear form $\langle \cdot, \cdot \rangle_\gamma$ on $L_r(\mathfrak{g})$ is $\mathfrak{g} \otimes I_\gamma$.*

Proof. It is clear from the definitions that $\mathfrak{g} \otimes I_\gamma$ is contained in the kernel. For convenience, let K denote the kernel. Then K is a \mathfrak{g} -submodule of $L_r(\mathfrak{g})$. Since \mathfrak{g} is a finite-dimensional simple Lie algebra (over \mathbb{C}), we have $K = \mathfrak{g} \otimes U$ for some subspace U of L_r . Choose $a, b \in \mathfrak{g}$ such that $\langle a, b \rangle = 1$. For any $u \in U$, $v \in L_r$, we have

$$0 = \langle a \otimes u, b \otimes v \rangle_\gamma = \langle a, b \rangle \gamma(uv) = \gamma(uv).$$

This implies $U \subset I_\gamma$, proving $K = \mathfrak{g} \otimes U \subset \mathfrak{g} \otimes I_\gamma$. \square

For a linear functional $\gamma : L_r \rightarrow \mathbb{C}$, set

$$G(r, \mathfrak{g}, \gamma) = \mathfrak{g} \otimes (L_r/I_\gamma) \quad (4.43)$$

and equip $G(r, \mathfrak{g}, \gamma)$ with the symmetric invariant bilinear form $\langle \cdot, \cdot \rangle_\gamma$, which is non-degenerate by Lemma 4.25. Denote by $\widehat{G}(r, \mathfrak{g}, \gamma)$ the affine Lie algebra. It is known that for any complex number ℓ , we have a simple \mathbb{Z} -graded vertex algebra $L_{\widehat{G}(r, \mathfrak{g}, \gamma)}(\ell, 0)$. Let $\psi : S_\mathfrak{c} \rightarrow L_r$ be a \mathbb{Z}^r -graded algebra homomorphism. Define $\gamma_\psi : L_r \rightarrow \mathbb{C}$ by $\mathfrak{t}^{\mathbf{m}} \mapsto \bar{\psi}(\mathfrak{c} \otimes \mathfrak{t}^{\mathbf{m}})$ for $\mathbf{m} \in \mathbb{Z}^r$. Then we have:

Lemma 4.26. *The simple \mathbb{Z} -graded vertex algebra $L^0(\psi, 0)$ is isomorphic to $L_{\widehat{G}(r, \mathfrak{g}, \gamma_\psi)}(\ell, 0)$ with $\ell = \gamma_\psi(1)$. In particular, $L^0(\psi, 0)_{(1)}$ is isomorphic to $G(r, \mathfrak{g}, \gamma_\psi)$.*

Proof. Recall that $L^0(\psi, 0)$ is an \mathbb{N} -graded vertex algebra with $L^0(\psi, 0)_{(0)} = \mathbb{C}\mathbf{1}$ and $L^0(\psi, 0)_{(1)}$ is linearly spanned by $a_{-1, \mathbf{m}}\mathbf{1}$ for $a \in \mathfrak{g}$, $\mathbf{m} \in \mathbb{Z}^r$. Note that $L^0(\psi, 0)_{(1)}$ is a Lie algebra. Let $\phi : L_r(\mathfrak{g}) \rightarrow L^0(\psi, 0)_{(1)}$ be the linear map given by $\phi(a \otimes \mathfrak{t}^{\mathbf{m}}) =$

$a_{-1,\mathbf{m}}\mathbf{1}$ for $a \in \mathfrak{g}$, $\mathbf{m} \in \mathbb{Z}^r$. This is a Lie algebra homomorphism. We show that $\ker \phi = \mathfrak{g} \otimes I_\gamma$. Let $a \in \mathfrak{g}$, $u \in I_\gamma$. For any $b \in \mathfrak{g}$, $v \in L_r$, we have

$$(b \otimes v)_1(a \otimes u)_{-1}\mathbf{1} = 0.$$

Then $U(\tau)(a \otimes u)_{-1}\mathbf{1}$ is a proper graded submodule of $L^0(\psi, 0)$. As $L^0(\psi, 0)$ is a \mathbb{Z} -graded irreducible τ -module, we must have $(a \otimes u)_{-1}\mathbf{1} = 0$. This shows that $\mathfrak{g} \otimes I_\gamma \subset \ker \phi$. Consequently, $L^0(\psi, 0)_{(1)}$ is isomorphic to $\mathfrak{g} \otimes (L_r/I_\gamma)$. \square

Let s be a positive integer, let $\mathbf{a}_1, \dots, \mathbf{a}_s \in (\mathbb{C}^\times)^r$ be distinct, and let ℓ_1, \dots, ℓ_s be complex numbers. To these data, we associate a \mathbb{Z}^r -graded algebra homomorphism $\psi : S_\mathfrak{c} \rightarrow L_r$ determined by

$$\psi(\mathbf{c} \otimes \mathbf{t}^{\mathbf{m}}) = \left(\sum_{i=1}^s \ell_i \mathbf{a}_i^{\mathbf{m}} \right) \mathbf{t}^{\mathbf{m}} \quad (4.44)$$

for $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r$, where $\mathbf{a}_i^{\mathbf{m}} = a_{i1}^{m_1} \cdots a_{ir}^{m_r}$. We have

$$\bar{\psi}(\mathbf{c} \otimes \mathbf{t}^{\mathbf{m}}) = \sum_{i=1}^s \ell_i \mathbf{a}_i^{\mathbf{m}}. \quad (4.45)$$

Remark 4.27. For each $1 \leq i \leq r$, suppose b_{i1}, \dots, b_{iN_i} are all the distinct elements of $\{a_{1i}, \dots, a_{si}\}$. Set

$$B = \{(b_{1j_1}, \dots, b_{rj_r}) \mid 1 \leq j_i \leq N_i \text{ for } 1 \leq i \leq r\} \subset (\mathbb{C}^\times)^r$$

with $|B| = N_1 N_2 \cdots N_r$. From definition we have $\{\mathbf{a}_1, \dots, \mathbf{a}_s\} \subset B$. Write $B = \{\mathbf{b}_1, \dots, \mathbf{b}_N\}$ with $N = N_1 N_2 \cdots N_r$ such that $\mathbf{b}_i = \mathbf{a}_i$ ($1 \leq i \leq s$), noticing that $\mathbf{a}_1, \dots, \mathbf{a}_s$ are distinct. Then $\bar{\psi}$ defined in (4.45) is a special case of the algebra homomorphism ψ defined in ([R2]; (3.14)). From [R2] (Lemma 3.11), the Lie algebra homomorphism $\Phi : L_r(\hat{\mathfrak{g}}) \rightarrow \hat{\mathfrak{g}}^{\oplus N}$, which is given by $a \otimes \mathbf{t}^{\mathbf{m}} \mapsto (\mathbf{b}_i^{\mathbf{m}} a)_{1 \leq i \leq N}$ ($a \in \hat{\mathfrak{g}}$), is surjective. Denote by π the projection map from $\hat{\mathfrak{g}}^{\oplus N}$ to $\hat{\mathfrak{g}}^{\oplus s}$ via $(a_i)_{1 \leq i \leq N} \mapsto (a_i)_{1 \leq i \leq s}$. Then $\pi \circ \Phi$ is surjective.

Note that for any complex number ℓ , one has a simple \mathbb{Z} -graded vertex algebra $L_{\hat{\mathfrak{g}}}(\ell, 0)$ associated to the affine Lie algebra $\hat{\mathfrak{g}}$. Furthermore, if $\ell \neq -h^\vee$, where h^\vee is the dual Coxeter number of \mathfrak{g} , $L_{\hat{\mathfrak{g}}}(\ell, 0)$ is a simple vertex operator algebra. We have:

Proposition 4.28. *Let s be a positive integer, let ℓ_1, \dots, ℓ_s be complex numbers, and let $\mathbf{a}_1, \dots, \mathbf{a}_s \in (\mathbb{C}^\times)^r$ be distinct. Let $\psi : S_\mathfrak{c} \rightarrow L_r$ be defined as in (4.44). Then there exists a vertex algebra isomorphism which is also a τ -module isomorphism*

$$\eta : L^0(\psi, 0) \rightarrow L_{\hat{\mathfrak{g}}}(\ell_1, 0) \otimes \cdots \otimes L_{\hat{\mathfrak{g}}}(\ell_s, 0), \quad (4.46)$$

such that

$$\eta(a_{-1,\mathbf{m}}\mathbf{1}) = \sum_{i=1}^s \mathbf{a}_i^{\mathbf{m}} \left(\mathbf{1} \otimes \cdots \otimes \underset{\substack{\uparrow \\ i\text{-th}}}{a_{-1}\mathbf{1}} \otimes \cdots \otimes \mathbf{1} \right) \quad (4.47)$$

for $a \in \mathfrak{g}$, $\mathbf{m} \in \mathbb{Z}^r$. Furthermore, if $\ell_i \neq -h^\vee$ for $1 \leq i \leq s$, $L^0(\psi, 0)$ is a simple vertex operator algebra.

Proof. From [R2] (Lemma 3.11) (see Remark 4.27), the right hand side is an irreducible \mathbb{Z} -graded τ -module with $\mathfrak{c} \otimes \mathbf{t}^{\mathbf{m}}$ acting as scalar

$$\sum_{i=1}^s \ell_i \mathbf{a}_i^{\mathbf{m}} \quad (= \bar{\psi}(\mathfrak{c} \otimes \mathbf{t}^{\mathbf{m}}))$$

and with $a \otimes t_0^k \mathbf{t}^{\mathbf{m}}$ acting as

$$\sum_{i=1}^s \ell_i \mathbf{a}_i^{\mathbf{m}} \pi_i(a \otimes t_0^k)$$

for $a \in \mathfrak{g}$, $(k, \mathbf{m}) \in \mathbb{Z} \times \mathbb{Z}^r$, where π_i is the Lie algebra embedding of $\hat{\mathfrak{g}}$ into $\hat{\mathfrak{g}}^{\oplus s}$. Let V denote this τ -module locally and let $\mathbf{1}_R$ denote the generator $\mathbf{1}^{\otimes s}$. We have

$$L_r(\mathfrak{b})\mathbf{1}_R = 0, \quad u \cdot \mathbf{1}_R = \bar{\psi}(u)\mathbf{1}_R \quad \text{for } u \in S_{\mathfrak{c}}.$$

It follows that there is a τ -module isomorphism η from $L^0(\psi, 0)$ to V with $\eta(\mathbf{1}) = \mathbf{1}_R$. For $a \in \mathfrak{g}$, $\mathbf{m} \in \mathbb{Z}^r$, we have

$$\eta(a_{-1,\mathbf{m}}\mathbf{1}) = a_{-1,\mathbf{m}}\mathbf{1}_R = \sum_{i=1}^s \mathbf{a}_i^{\mathbf{m}} \left(\mathbf{1} \otimes \cdots \otimes \underset{\substack{\uparrow \\ i\text{-th}}}{a_{-1}\mathbf{1}} \otimes \cdots \otimes \mathbf{1} \right).$$

By (2.13) we have

$$Y(a_{-1,\mathbf{m}}\mathbf{1}; x_0, \mathbf{x}) = \sum_{m_0 \in \mathbb{Z}} a_{m_0, \mathbf{m}} x_0^{-m_0-1} \mathbf{x}^{-\mathbf{m}},$$

so that

$$Y(a_{-1,\mathbf{m}}\mathbf{1}, x_0) = Y(a_{-1,\mathbf{m}}\mathbf{1}; x_0, \mathbf{x})|_{\mathbf{x}=1} = \sum_{m_0 \in \mathbb{Z}} a_{m_0, \mathbf{m}} x_0^{-m_0-1} = a(x_0, \mathbf{m}).$$

On the other hand, for $a \in \mathfrak{g}$, $\mathbf{m} \in \mathbb{Z}^r$, we have

$$Y(\eta(a_{-1,\mathbf{m}}\mathbf{1}), x_0) = \sum_{i=1}^s \mathbf{a}_i^{\mathbf{m}} Y(\pi_i(a), x_0) = \rho_R(a(x_0, \mathbf{m})),$$

where ρ_R is the Lie algebra homomorphism affording the τ -module V and π_i is the Lie algebra embedding of \mathfrak{g} into $\oplus_{j=1}^s \mathfrak{g}$. With η a τ -module homomorphism, we have

$$\eta(Y(u, x_0)w) = Y(\eta(u), x_0)\eta(w) \quad \text{for all } u \in L^0(\psi, 0)_{(1)}, w \in L^0(\psi, 0).$$

As $L^0(\psi, 0)_{(1)}$ generates $L^0(\psi, 0)$ as a vertex algebra, it follows that η is a homomorphism of vertex algebras. Thus η is an isomorphism of vertex algebras. Restricted onto $L^0(\psi, 0)_{(1)}$, η becomes an isomorphism of Lie algebras onto $\oplus_{i=1}^s \mathfrak{g}$. \square

Note (see [FZ], [DL], [Li1], [MP1], [MP2]) that for every positive integer ℓ , the module category of $L_{\hat{\mathfrak{g}}}(\ell, 0)$ is semi-simple and irreducible $L_{\hat{\mathfrak{g}}}(\ell, 0)$ -modules are exactly integrable highest weight $\hat{\mathfrak{g}}$ -modules of level ℓ . From [FHL], [DLM] (Proposition 3.3) and Proposition 4.28, we immediately have:

Corollary 4.29. *Let $\psi : S_c \rightarrow L_r$ be the \mathbb{Z}^r -graded algebra homomorphism given as in Proposition 4.28. Assume that ℓ_1, \dots, ℓ_s are positive integers. Then every $L^0(\psi, 0)$ -module is completely reducible and $L^0(\psi, 0)$ has only finitely many irreducible modules up to isomorphism.*

5 Integrability

In this section, we give a necessary and sufficient condition that $L(\psi, 0)$ is an integrable τ -module, where $\psi : S_c \rightarrow L_r$ is a \mathbb{Z}^r -graded algebra homomorphism such that $\text{Im} \psi$ is a \mathbb{Z}^r -graded simple S_c -module.

Fix a Cartan subalgebra \mathfrak{h} of \mathfrak{g} and denote by Δ and Δ_+ the sets of roots and positive roots of \mathfrak{g} , respectively. Set

$$\mathfrak{g}_{\pm} = \sum_{\pm \alpha \in \Delta_+} \mathfrak{g}_{\alpha}.$$

Let θ be the highest root of \mathfrak{g} and we fix nonzero root vectors $e_{\theta} \in \mathfrak{g}_{\theta}$, $f_{\theta} \in \mathfrak{g}_{-\theta}$ such that $\langle e_{\theta}, f_{\theta} \rangle = 1$. Let A be any commutative associative algebra with identity (over \mathbb{C}). A $\hat{\mathfrak{g}} \otimes A$ -module W is said to be *integrable* if \mathfrak{h} acts semisimply and $\mathfrak{g}_{\alpha} \otimes t_0^k u$ acts locally nilpotently on W for any $\alpha \in \Delta$, $k \in \mathbb{Z}$, $u \in A$.

Recall that $L(\psi, 0)$ and $L^0(\psi, 0)$ are naturally restricted τ -modules, where $L^0(\psi, 0)$ is an irreducible quotient τ -module of $L(\psi, 0)$. We have:

Lemma 5.1. *$L(\psi, 0)$ is an integrable τ -module if and only if $L^0(\psi, 0)$ is integrable.*

Proof. Since $L^0(\psi, 0)$ is a quotient τ -module of $L(\psi, 0)$, the “only if” part is clear. Let π be the natural quotient map from $L(\psi, 0)$ to $L^0(\psi, 0)$. By Lemma 3.9, we have an injective $(r+1)$ -toroidal vertex algebra homomorphism $\tilde{\pi}$ from $L(\psi, 0)$ to $L_r(L^0(\psi, 0))$. As $\tilde{\pi}$ is injective, $L(\psi, 0)$ is a τ -submodule of $L_r(L^0(\psi, 0))$. It can be readily seen that $L_r(L^0(\psi, 0))$ is an integrable τ -module if and only if $L^0(\psi, 0)$ is an integrable τ -module. Then the “if” part follows. \square

Let $\{e, f, h\}$ be the standard basis of \mathfrak{sl}_2 . Consider the r -loop algebra $L_r(\mathfrak{sl}_2)$. We equip $L_r(\mathfrak{sl}_2)$ with the \mathbb{Z} -grading given by

$$\deg L_r(\mathbb{C}e) = 1, \quad \deg(L_r(\mathbb{C}f)) = -1, \quad \deg(L_r(\mathbb{C}h)) = 0,$$

to make $L_r(\mathfrak{sl}_2)$ a \mathbb{Z} -graded Lie algebra. Notice that Lie algebra $L_r(\mathbb{C}e \oplus \mathbb{C}h)$ is the semi-product of abelian subalgebra $L_r(\mathbb{C}h)$ with ideal $L_r(\mathbb{C}e)$.

Let $\phi : L_r(\mathbb{C}h) \rightarrow \mathbb{C}$ be a linear function. We have a 1-dimensional $L_r(\mathbb{C}e \oplus \mathbb{C}h)$ -module, denoted by \mathbb{C}_ϕ , where $\mathbb{C}_\phi = \mathbb{C}$ and

$$(h \otimes \mathbf{t}^{\mathbf{m}}) \cdot 1 = \phi(h \otimes \mathbf{t}^{\mathbf{m}})1, \quad (e \otimes \mathbf{t}^{\mathbf{m}}) \cdot 1 = 0 \quad \text{for } \mathbf{m} \in \mathbb{Z}^r. \quad (5.1)$$

Form an induced module

$$U(\phi) = U(L_r(\mathfrak{sl}_2)) \otimes_{U(L_r(\mathbb{C}e \oplus \mathbb{C}h))} \mathbb{C}_\phi. \quad (5.2)$$

Let 1_ϕ denote the generator $1 \otimes 1$. From the P-B-W theorem we have $U(\phi) = U(L_r(\mathbb{C}f))1_\phi$. It then follows that h acts semisimply on $U(\phi)$ with

$$U(\phi) = \bigoplus_{n \in \mathbb{N}} U(\phi)_{\phi(h) - 2n},$$

where $U(\phi)_\lambda = \{u \in U(\phi) \mid hu = \lambda u\}$ for $\lambda \in \mathbb{C}$. Consequently, U_ϕ has a unique maximal $L_r(\mathfrak{sl}_2)$ -submodule. Denote by L_ϕ the unique simple quotient $L_r(\mathfrak{sl}_2)$ -module of $U(\phi)$.

Lemma 5.2. *Let I be the sum of ideals J of L_r such that $\phi(h \otimes J) = 0$. Then, for $a \in L_r$, $\phi(h \otimes aL_r) = 0$ if and only if $a \in I$, and if and only if $(f \otimes a)1_\phi = 0$. On the other hand, $\dim(L_r/I) < \infty$ if and only if $\dim(L_r/\sqrt{I}) < \infty$, where*

$$\sqrt{I} = \{a \in L_r \mid a^n \in I \text{ for some positive integer } n\}. \quad (5.3)$$

Proof. The first part is clear as aL_r is an ideal containing a . If $(f \otimes a)1_\phi = 0$, for any $b \in L_r$ we have

$$\phi(h \otimes ab)1_\phi = [e \otimes b, f \otimes a]1_\phi = (e \otimes b)(f \otimes a)1_\phi - (f \otimes a)(e \otimes b)1_\phi = 0,$$

which implies $aL_r \subset I$, and hence $a \in I$. Conversely, assume $a \in I$. We have

$$(e \otimes b)(f \otimes a)1_\phi = (f \otimes a)(e \otimes b)1_\phi + \phi(h \otimes ab)1_\phi = 0$$

for all $b \in L_r$. That is, $L_r(\mathbb{C}e)(f \otimes a)1_\phi = 0$. As L_ϕ is irreducible, we must have $(f \otimes a)1_\phi = 0$.

As for the last assertion, assume $\dim(L_r/\sqrt{I}) < \infty$. As $\dim(\mathbb{C}[t_i] + \sqrt{I})/\sqrt{I} < \infty$ for each $1 \leq i \leq r$, there are nonzero polynomials $p_1(t), \dots, p_r(t) \in \mathbb{C}[t]$ such that

$$(p_1(t_1), \dots, p_r(t_r)) \subset \sqrt{I}.$$

Then there exists a positive integer n such that

$$(p_1(t_1)^n, \dots, p_r(t_r)^n) \subset I.$$

Thus, $\dim(L_r/I) < \infty$. The other direction is clear as $I \subset \sqrt{I}$. □

We have:

Proposition 5.3. L_ϕ is an integrable $L_r(\mathfrak{sl}_2)$ -module if and only if $\dim L_\phi < \infty$.

Proof. For any $\mathbf{m} \in \mathbb{Z}^r$, set

$$\mathfrak{sl}_2^{(\mathbf{m})} = \mathbb{C}(e \otimes \mathbf{t}^{\mathbf{m}}) + \mathbb{C}(f \otimes \mathbf{t}^{-\mathbf{m}}) + \mathbb{C}h,$$

which is canonically isomorphic to \mathfrak{sl}_2 . It can be readily seen that L_ϕ is integrable if $\dim L_\phi < \infty$.

Now, we assume that L_ϕ is integrable. Recall from Lemma 5.2 the ideal I of L_r . We next prove that I is co-finite dimensional. By Lemma 5.2, we need to prove that \sqrt{I} is co-finite dimensional. Note that

$$\sqrt{I} = \bigcap_{i=1}^d Q_i,$$

where Q_i for $1 \leq i \leq d$ are prime ideals of L_r , so that L_r/Q_i are integral ring extensions over \mathbb{C} .

We first consider the case that all L_r/Q_i are finite dimensional. In this case, every L_r/Q_i is a finite-dimensional field extension of \mathbb{C} , which must be 1-dimensional. Thus $Q_i = (t_1 - c_{i1}, \dots, t_r - c_{ir})$ for some non-zero complex numbers c_{i1}, \dots, c_{ir} , for $1 \leq i \leq d$. Consequently,

$$\left(\prod_{i=1}^d (t_1 - c_{i1}), \dots, \prod_{i=1}^d (t_r - c_{ir}) \right) \subset \bigcap_{i=1}^d Q_i = \sqrt{I}.$$

Thus \sqrt{I} is co-finite dimensional, and so is I .

Now, we prove that all L_r/Q_i must be finite dimensional. Otherwise, L_r/Q_i is infinite dimensional for some i . As L_r/Q_i is also an infinite dimensional integral domain (over \mathbb{C}), there exists $y \in L_r$ such that $y + Q_i$ is a transcendental element of L_r/Q_i over \mathbb{C} . Then y^n for $n \in \mathbb{N}$ are linearly independent in L_r modulo Q_i , which implies that y^n for $n \in \mathbb{N}$ are linearly independent in L_r modulo I . By Lemma 5.2, $(f \otimes y^n)1$ with $n \in \mathbb{N}$ are linearly independent in L_ϕ . For each positive integer n , we set

$$L(n) = \bigoplus_{0 \leq m < n} \mathbb{C}(f \otimes y^m)1_\phi \subset L_\phi.$$

We claim that for each positive integer n and for any positive integers i_1, \dots, i_n ,

$$(e \otimes 1)^{n-1} \prod_{j=1}^n (f \otimes y^{i_j})1_\phi \equiv c_n (f \otimes y^N)1_\phi \quad \text{modulo } L(N), \quad (5.4)$$

where $N = i_1 + \dots + i_n$ and $c_n = (-1)^{n-1}n!(n-1)!$. It is clear that (5.4) is true when $n = 1$. Assume that n is a positive integer such that (5.4) holds for each

positive integer less than or equal to n . For positive integers i_1, \dots, i_{n+1} and any $1 \leq k < \ell \leq n+1$, we set $N' = \sum_{j=1}^{n+1} i_j$ and set

$$A(k) = \prod_{\substack{1 \leq j \leq n+1 \\ j \neq k}} (f \otimes y^{i_j}) \quad \text{and} \quad A(k, \ell) = \prod_{\substack{1 \leq j \leq n+1 \\ j \neq k, \ell}} (f \otimes y^{i_j}).$$

Then

$$\begin{aligned} (e \otimes 1)^n \prod_{j=1}^{n+1} (f \otimes y^{i_j}) 1_\phi &\equiv (e \otimes 1)^{n-1} \left[e \otimes 1, \prod_{j=1}^n (f \otimes y^{i_j}) \right] 1_\phi \\ &\equiv (e \otimes 1)^{n-1} \sum_{k=1}^{n+1} \varphi(h \otimes y^{i_k}) A(k) 1_\phi - 2(e \otimes 1)^{n-1} \sum_{1 \leq k < \ell \leq n+1} A(k, \ell) (f \otimes y^{i_k + i_\ell}) 1_\phi \\ &\equiv c_n \sum_{k=1}^{n+1} \varphi(h \otimes y^{i_k}) (f \otimes y^{M_k}) 1_\phi - 2c_n \sum_{1 \leq k < \ell \leq n+1} (f \otimes y^{N'}) 1_\phi \quad \text{modulo } L(N''), \end{aligned}$$

where $M_k = \sum_{j \neq k} i_j < N'$ and $L(N'') = \sum_{k=1}^{n+1} L(M_k) + L(N') = L(N')$. Therefore,

$$\begin{aligned} (e \otimes 1)^n \prod_{j=1}^{n+1} (f \otimes y^{i_j}) 1_\phi &\equiv -c_n(n+1)n (f \otimes y^{N'}) 1_\phi \\ &\equiv c_{n+1} (f \otimes y^{N'}) 1_\phi \quad \text{modulo } L(N'). \end{aligned}$$

This proves that (5.4) holds for all positive integers n .

As L_ϕ is integrable, there exists a positive integer n such that $(f \otimes y)^n 1_\phi = 0$. Then

$$0 \equiv (e \otimes 1)^{n-1} (f \otimes y)^n 1_\phi \equiv c_n (f \otimes y^n) 1_\phi \neq 0 \quad \text{modulo } L(n).$$

This contradiction implies that all L_r/Q_i must be finite dimensional. Therefore, I is co-finite dimensional. With $U(L_r(\mathbb{C}f))$ commutative, we have

$$L_\phi = U(L_r(\mathfrak{sl}_2)) 1_\phi = U(L_r(\mathbb{C}f)) 1_\phi = U(\mathbb{C}f \otimes (L_r/I)) 1_\phi.$$

Since $\dim(L_r/I) < \infty$ and for every $\mathbf{m} \in \mathbb{Z}^r$, $f \otimes \mathbf{t}^{\mathbf{m}}$ is nilpotent on 1_ϕ , it follows that L_ϕ is finite dimensional. \square

Now, we have:

Theorem 5.4. *Let $\psi : S_\zeta \rightarrow L_r$ be a \mathbb{Z}^r -graded algebra homomorphism such that $\text{Im} \psi$ is a \mathbb{Z}^r -graded simple S_ζ -module. Then $L(\psi, 0)$ is an integrable τ -module if*

and only if there exist finitely many positive integers ℓ_1, \dots, ℓ_s and distinct vectors $\mathbf{a}_1, \dots, \mathbf{a}_s \in (\mathbb{C}^\times)^r$ such that

$$\psi(\mathbf{c} \otimes \mathbf{t}^{\mathbf{m}}) = \left(\sum_{i=1}^s \ell_i \mathbf{a}_i^{\mathbf{m}} \right) \mathbf{t}^{\mathbf{m}} \quad \text{for } \mathbf{m} \in \mathbb{Z}^r.$$

Proof. The “if” part follows from Proposition 4.28 and Lemma 5.1. Now, we assume that $L(\psi, 0)$ is integrable. Then $L^0(\psi, 0)$ is integrable by Lemma 5.1. We fix an $\alpha \in \Delta$ and choose $e \in \mathfrak{g}_\alpha$, $f \in \mathfrak{g}_{-\alpha}$ such that $[e, f] = \check{\alpha}$ and $\langle e, f \rangle = 1$. Set

$$\mathfrak{s} = \mathbb{C}(e \otimes t_0) + \mathbb{C}(f \otimes t_0^{-1}) + \mathbb{C}(\check{\alpha} + \mathbf{c}).$$

Notice that $\mathfrak{s} \cong \mathfrak{sl}_2$. Then $U(L_r(\mathfrak{s}))\mathbf{1}$ is an integrable $L_r(\mathfrak{s})$ -submodule of $L^0(\psi, 0)$. Define a \mathbb{Z}^r -graded algebra homomorphism $\psi_\alpha : U(L_r(\check{\alpha} + \mathbf{c})) \rightarrow L_r$ by

$$\psi_\alpha((\check{\alpha} + \mathbf{c}) \otimes \mathbf{t}^{\mathbf{m}}) = \psi(\mathbf{c} \otimes \mathbf{t}^{\mathbf{m}}) \quad \text{for } \mathbf{m} \in \mathbb{Z}^r.$$

Then $\text{Im} \psi_\alpha$ is a \mathbb{Z}^r -graded simple $U(L_r(\check{\alpha} + \mathbf{c}))$ -module. Set $\phi = E(1) \circ \psi_\alpha$. We have $L_r(e \otimes t_0)\mathbf{1} = 0$ and $X \cdot \mathbf{1} = \phi(X)\mathbf{1}$ for $X \in U(L_r(\check{\alpha} + \mathbf{c}))$, noticing that $(\check{\alpha} \otimes \mathbf{t}^{\mathbf{m}}) \cdot \mathbf{1} = 0$ for $\mathbf{m} \in \mathbb{Z}^r$. It follows that L_ϕ is a homomorphism image of $U(L_r(\mathfrak{s}))\mathbf{1}$. Consequently, L_ϕ is integrable. By Proposition 5.3, L_ϕ is finite dimensional. From the proof of Proposition 3.20 in [R2], there exist finitely many nonzero dominant integral weights $\lambda_1, \dots, \lambda_s$ of \mathfrak{s} and distinct vectors $\mathbf{a}_1, \dots, \mathbf{a}_s \in (\mathbb{C}^\times)^r$ such that

$$\phi((\check{\alpha} + \mathbf{c}) \otimes \mathbf{t}^{\mathbf{m}}) = \sum_{i=1}^s \mathbf{a}_i^{\mathbf{m}} \lambda_i(\check{\alpha} + \mathbf{c})$$

for $\mathbf{m} \in \mathbb{Z}^r$. Thus

$$\psi(\mathbf{c} \otimes \mathbf{t}^{\mathbf{m}}) = \left(\sum_{i=1}^s \ell_i \mathbf{a}_i^{\mathbf{m}} \right) \mathbf{t}^{\mathbf{m}} \quad \text{for } \mathbf{m} \in \mathbb{Z}^r,$$

where $\ell_i = \lambda_i(\check{\alpha} + \mathbf{c})$. □

6 Irreducible $L(\psi, 0)$ -modules

In this section, we determine irreducible $L(\psi, 0)$ -modules with $L(\psi, 0)$ an integrable τ -module.

For this section, we assume that $\psi : S_{\mathbf{c}} \rightarrow L_r$ is the \mathbb{Z}^r -graded algebra homomorphism defined in (4.44), associated to positive integers ℓ_1, \dots, ℓ_s and distinct vectors

$$\mathbf{a}_i = (a_{i1}, \dots, a_{ir}) \in (\mathbb{C}^\times)^r \tag{6.1}$$

for $1 \leq i \leq s$. We next present some basic results about $\text{Im} \psi$. First, we have:

Lemma 6.1. *There exist positive integers ν_1, \dots, ν_r such that*

$$\mathbb{C}[t_1^{\pm\nu_1}, \dots, t_r^{\pm\nu_r}] \subset \text{Im}\psi. \quad (6.2)$$

Proof. Note that $\text{Im}\psi$ is a subalgebra of L_r . Then it suffices to prove that for each $1 \leq j \leq r$, there exists a positive integer ν_j such that $t_j^{\pm\nu_j} \in \text{Im}\psi$. Furthermore, it suffices to prove that for each $1 \leq j \leq r$, there exist a positive integer p_j and a negative integer n_j such that $t_j^{p_j}, t_j^{n_j} \in \text{Im}\psi$. Fix $1 \leq j \leq r$. Recall that $\psi(\mathbf{c} \otimes t_j^m) = \left(\sum_{i=1}^s \ell_i a_{ij}^m \right) t_j^m$ for $m \in \mathbb{Z}$. Then it suffices to prove that $\sum_{i=1}^s \ell_i a_{ij}^m \neq 0$ for some positive integer m and for some negative integer m . Let $S_1 \cup \dots \cup S_p$ be the partition of the set $\{1, 2, \dots, s\}$ corresponding to the equivalence relation given by $\mu \equiv \nu$ if and only if $a_{\mu,j} = a_{\nu,j}$. For $1 \leq q \leq p$, set $b_q = a_{i,j}$ for some $i \in S_q$. Then

$$\sum_{i=1}^s \ell_i a_{ij}^m = \left(\sum_{i \in S_1} \ell_i \right) b_1^m + \dots + \left(\sum_{i \in S_p} \ell_i \right) b_p^m.$$

Since b_1, \dots, b_p are distinct nonzero complex numbers, the system of linear equations

$$b_1^m x_1 + \dots + b_p^m x_p = 0$$

for $m = 1, 2, \dots$ does not have nontrivial solutions. As ℓ_1, \dots, ℓ_s are positive integers, we must have that $\sum_{i=1}^s \ell_i a_{ij}^m \neq 0$ for some positive integer m . It is also clear that $\sum_{i=1}^s \ell_i a_{ij}^m \neq 0$ for some negative integer m . \square

As $\text{Im}\psi$ is a \mathbb{Z}^r -graded simple subalgebra of L_r , we immediately have:

Corollary 6.2. *Set*

$$\Lambda(\psi) = \{\mathbf{m} \in \mathbb{Z}^r \mid \mathbf{t}^{\mathbf{m}} \in \text{Im}\psi\}. \quad (6.3)$$

Then $\Lambda(\psi)$ is a cofinite subgroup of \mathbb{Z}^r .

Definition 6.3. For each $1 \leq j \leq r$, let ν_j be the least positive integer such that $t_j^{\pm\nu_j} \in \text{Im}\psi$ (see Lemma 6.1). Furthermore, for each $1 \leq j \leq r$, set

$$S_j = \{a_{ij}^{\nu_j} \mid i = 1, 2, \dots, s\}$$

(multiplicity-free) and define

$$p_j(t_j) = \prod_{c \in S_j} (t_j^{\nu_j} - c). \quad (6.4)$$

From definition we have

$$p_j(t_j) \in \mathbb{C} [t_1^{\pm\nu_1}, \dots, t_r^{\pm\nu_r}] \subset \text{Im}\psi. \quad (6.5)$$

Remark 6.4. Note that any $V(S_{\mathfrak{c}}, 0)$ -module W is naturally an $S_{\mathfrak{c}}$ -module and $\ker \psi \subset S_{\mathfrak{c}}$. We have that a $V(\psi, 0)$ -module amounts to a $V(S_{\mathfrak{c}}, 0)$ -module W such that $(\ker \psi)W = 0$. On the other hand, with $S_{\mathfrak{c}}$ a subalgebra of $U(\tau)$, every τ -module W is also naturally an $S_{\mathfrak{c}}$ -module, and furthermore, if $(\ker \psi)W = 0$, then W is naturally an $(\text{Im}\psi)$ -module as $S_{\mathfrak{c}}/\ker \psi \cong \text{Im}\psi$.

Remark 6.5. We here present a simple fact about integrable $\hat{\mathfrak{g}}$ -modules. Let W be a nonzero restricted $\hat{\mathfrak{g}}$ -module of level ℓ . Let θ be the highest root of \mathfrak{g} and let e_{θ} be a non-zero vector in \mathfrak{g}_{θ} . Assume that $e_{\theta}(x)^k = 0$ for some positive integer k . Then we show that W is an integrable $\hat{\mathfrak{g}}$ -module as follows: (1) It is known that W is naturally a module for vertex algebra $V_{\hat{\mathfrak{g}}}(\ell, 0)$ (cf. [Li1]). (2) Let J be the annihilating ideal of W and set $\overline{V} = V_{\hat{\mathfrak{g}}}(\ell, 0)/J$. Then W is a faithful \overline{V} -module. As $Y_W(e_{\theta}(-1)^k \mathbf{1}, x) = Y_W(e_{\theta}, x)^k = e_{\theta}(x)^k = 0$, we have $e_{\theta}(-1)^k \mathbf{1} = 0$ in \overline{V} . Then \overline{V} is an integrable $\hat{\mathfrak{g}}$ -module (by using the Chevalley generators). (3) As $W \neq 0$, we must have $\overline{V} \neq 0$. It follows that $\overline{V} = L_{\hat{\mathfrak{g}}}(\ell, 0)$, which implies that $L_{\hat{\mathfrak{g}}}(\ell, 0)$ is integrable and ℓ must be a nonnegative integer (see [K]). (4) As W is a weak $L_{\hat{\mathfrak{g}}}(\ell, 0)$ -module, from [DLM] W must be an integrable $\hat{\mathfrak{g}}$ -module.

The following is a characterization of $L^0(\psi, 0)$ as a quotient module of $V^0(\psi, 0)$:

Proposition 6.6. *Set $\ell = \ell_1 + \dots + \ell_s$. Then $L^0(\psi, 0) = V^0(\psi, 0)/I$, where I is the ideal of vertex algebra $V^0(\psi, 0)$ generated by*

$$e_{\theta}(-1, \mathbf{m})^{\ell+1} \mathbf{1}, \quad (u \otimes t_0^{-1} f(\mathbf{t})) \mathbf{1} \quad (6.6)$$

for $\mathbf{m} \in \mathbb{Z}^r$, $u \in \mathfrak{g}$, $f(\mathbf{t}) \in p_1(t_1)L_r + \dots + p_r(t_r)L_r$.

Proof. Let π denote the quotient map from $V^0(\psi, 0)$ to $L^0(\psi, 0)$, which is also a τ -module homomorphism. Recall from Proposition 4.28 the explicit construction of $L^0(\psi, 0)$ in terms of highest weight irreducible $\hat{\mathfrak{g}}$ -modules $L_{\hat{\mathfrak{g}}}(\ell_i, 0)$ for $i = 1, \dots, s$. Let Δ_i denote the natural algebra map from $U(\hat{\mathfrak{g}})$ to $U(\hat{\mathfrak{g}})^{\otimes s}$. We have

$$\pi(e_{\theta}(-1, \mathbf{m})^{\ell+1} \mathbf{1}) = e_{\theta}(-1, \mathbf{m})^{\ell+1} \pi(\mathbf{1}) = \left(\sum_{i=1}^s \mathbf{a}_i^{\mathbf{m}} \Delta_i(e_{\theta}(-1)) \right)^{\ell+1} \pi(\mathbf{1}) = 0$$

as $\Delta_i(e_{\theta}(-1))^{\ell_i+1} \pi(\mathbf{1}) = 0$ for $1 \leq i \leq s$. On the other hand, for $u \in \mathfrak{g}$, as

$$u(x_0, \mathbf{x}) \pi(\mathbf{1}) = \sum_{i=1}^s \delta\left(\frac{\mathbf{a}_i}{\mathbf{x}}\right) \Delta_i(u(x_0)) \pi(\mathbf{1}),$$

where $\delta\left(\frac{\mathbf{a}_i}{\mathbf{x}}\right) = \prod_{j=1}^r \delta\left(\frac{a_{ij}}{x_j}\right)$, we have

$$p_j(x_j) u(x_0, \mathbf{x}) \pi(\mathbf{1}) = \sum_{i=1}^s p_j(x_j) \delta\left(\frac{\mathbf{a}_i}{\mathbf{x}}\right) \Delta_i(u(x_0)) \pi(\mathbf{1}) = 0$$

for $1 \leq j \leq r$. Thus for any $f(\mathbf{t}) \in p_1(t_1)L_r + \cdots + p_r(t_r)L_r$, we have $f(\mathbf{x})u(x_0, \mathbf{x})\pi(\mathbf{1}) = 0$, which implies $(u \otimes t_0^{n_0} f(\mathbf{t}))\pi(\mathbf{1}) = 0$ for all $n_0 \in \mathbb{Z}$. It now follows that $I \subset \ker \pi$.

From definition, the following relations hold in $V^0(\psi, 0)/I$:

$$\begin{aligned} Y^0(e_\theta(-1, \mathbf{m})\mathbf{1}, x_0)^{\ell+1} &= Y^0(e_\theta(-1, \mathbf{m})^{\ell+1}\mathbf{1}, x_0) = 0, \\ \sum_{k \in \mathbb{Z}} u \otimes t_0^k f(\mathbf{t}) x_0^{-k-1} &= Y^0((u \otimes t_0^{-1} f(\mathbf{t}))\mathbf{1}, x_0) = 0 \end{aligned}$$

for $\mathbf{m} \in \mathbb{Z}^r$, $u \in \mathfrak{g}$, $f(\mathbf{t}) \in p_1(t_1)L_r + \cdots + p_r(t_r)L_r$. Set

$$K = L_r / (p_1(t_1)L_r + \cdots + p_r(t_r)L_r).$$

The second relation implies that $V^0(\psi, 0)/I$ is a restricted module for the quotient algebra $\hat{\mathfrak{g}} \otimes K$ of τ . As $K \cong \mathbb{C}^{\oplus N}$ with $N = \prod_{j=1}^s \deg p_j(t_j)$,

$$\hat{\mathfrak{g}} \otimes K \cong \bigoplus_{N\text{-copies}} \hat{\mathfrak{g}}$$

canonically. It then follows from the first relation that $V^0(\psi, 0)/I$ is an integrable module for the quotient algebra (see Remark 6.5) and then for τ . Then $V^0(\psi, 0)/I$ is a completely reducible module for the quotient algebra (and then for τ). Since $V^0(\psi, 0)/I$, a nonzero τ -module, is generated by $\mathbf{1}$ (a highest weight vector), it follows that $V^0(\psi, 0)/I$ is irreducible. Consequently, $V^0(\psi, 0)/I = L^0(\psi, 0)$. \square

Next, we shall determine $L(\psi, 0)$ and its irreducible modules in terms of $L^0(\psi, 0)$ and its irreducible modules. Recall that $L^0(\psi, 0)$ is the \mathbb{Z} -graded simple quotient vertex algebra of $L(\psi, 0)$, $V^0(\psi, 0)$ is the quotient vertex algebra of $V(\psi, 0)$ modulo the ideal generated by $\mathbf{t}^{\mathbf{m}} - \mathbf{1}$ for $\mathbf{m} \in \mathbb{Z}^r$ with $\mathbf{t}^{\mathbf{m}} \in \text{Im} \psi$. We have the following commutative diagram

$$\begin{array}{ccc} V(\psi, 0) & \xrightarrow{\gamma} & V^0(\psi, 0) \\ \downarrow \alpha & & \downarrow \alpha^0 \\ L(\psi, 0) & \xrightarrow{\pi} & L^0(\psi, 0). \end{array} \quad (6.7)$$

Note that α, α^0, γ and π are all canonical quotient maps, which are homomorphisms of vertex algebras and τ -modules. Furthermore, using Lemma 3.9 we obtain the following commutative diagram

$$\begin{array}{ccc} V(\psi, 0) & \xrightarrow{\tilde{\gamma}} & L_r(V^0(\psi, 0)) \\ \downarrow \alpha & & \downarrow \alpha^0 \otimes 1 \\ L(\psi, 0) & \xrightarrow{\tilde{\pi}} & L_r(L^0(\psi, 0)). \end{array} \quad (6.8)$$

From Lemma 3.9, $\tilde{\pi}$ is injective.

Note that

$$V(\psi, 0) \cong U(L_r(\mathfrak{g} \otimes t_0^{-1}\mathbb{C}[t_0^{-1}])) \otimes \text{Im}\psi, \quad (6.9)$$

$$V^0(\psi, 0) \cong U(L_r(\mathfrak{g} \otimes t_0^{-1}\mathbb{C}[t_0^{-1}])) \quad (6.10)$$

as vector spaces. In view of this, we shall always view $U(L_r(\mathfrak{g} \otimes t_0^{-1}\mathbb{C}[t_0^{-1}]))$ as a subspace of $V(\psi, 0)$ and $V^0(\psi, 0)$. Since $L_r(\mathfrak{g} \otimes t_0^{-1}\mathbb{C}[t_0^{-1}])$ is a \mathbb{Z}^r -graded Lie algebra, $U(L_r(\mathfrak{g} \otimes t_0^{-1}\mathbb{C}[t_0^{-1}]))$ is an associative \mathbb{Z}^r -graded algebra. For $\mathbf{n} \in \mathbb{Z}^r$, denote by $U(\mathbf{n})$ the degree \mathbf{n} subspace of $U(L_r(\mathfrak{g} \otimes t_0^{-1}\mathbb{C}[t_0^{-1}]))$.

We have

$$\gamma(X \otimes \mathbf{t}^{\mathbf{m}}) = X \quad \text{for } X \in U(L_r(\mathfrak{g} \otimes t_0^{-1}\mathbb{C}[t_0^{-1}))), \mathbf{m} \in \Lambda(\psi). \quad (6.11)$$

For the homomorphism $\tilde{\gamma} : V(\psi, 0) \rightarrow L_r(V^0(\psi, 0))$, we have

$$\tilde{\gamma}(X \otimes \mathbf{t}^{\mathbf{m}}) = X \otimes \mathbf{t}^{\mathbf{m}+\mathbf{n}} \quad (6.12)$$

for $X \in U(\mathbf{n})$, $\mathbf{m} \in \Lambda(\psi)$ with $\mathbf{n} \in \mathbb{Z}^r$. Furthermore, we have:

Lemma 6.7. *The map $\tilde{\gamma}$ is injective. On the other hand, we have*

$$\text{Im}\tilde{\gamma} = \text{span} \{X \otimes \mathbf{t}^{\mathbf{m}} \mid X \in U(\mathbf{n}), \mathbf{m}, \mathbf{n} \in \mathbb{Z}^r \text{ with } \mathbf{m} - \mathbf{n} \in \Lambda(\psi)\}. \quad (6.13)$$

Proof. Since $\tilde{\gamma}$ preserves the \mathbb{Z}^r -grading, it suffices to show that $\tilde{\gamma}(X) \neq 0$ for any nonzero homogeneous vector $X \in V(\psi, 0)$. Assume $\deg X = \mathbf{m}$. Then

$$X = X_1 \otimes \mathbf{t}^{\mathbf{m}^1} + \cdots + X_k \otimes \mathbf{t}^{\mathbf{m}^k},$$

where $\mathbf{m}^1, \dots, \mathbf{m}^k \in \Lambda(\psi)$ are distinct and X_1, \dots, X_k are nonzero homogeneous elements of $U(L_r(\mathfrak{g} \otimes t_0^{-1}\mathbb{C}[t_0^{-1}]))$ of degrees $\mathbf{n}^1, \dots, \mathbf{n}^k$ such that $\mathbf{m}^i + \mathbf{n}^i = \mathbf{m}$. As $\mathbf{m}^1, \dots, \mathbf{m}^k$ are distinct, $\mathbf{n}^1, \dots, \mathbf{n}^k$ must be distinct, so that X_1, \dots, X_k are linearly independent. Thus

$$\tilde{\gamma}(X) = \tilde{\gamma}(X_1 \otimes \mathbf{t}^{\mathbf{m}^1} + \cdots + X_k \otimes \mathbf{t}^{\mathbf{m}^k}) = (X_1 + \cdots + X_k) \otimes \mathbf{t}^{\mathbf{m}} \neq 0.$$

This shows that $\tilde{\gamma}$ is injective.

Note that $X \otimes \mathbf{t}^{\mathbf{m}} \in \text{Im}\tilde{\gamma}$ for any $X \in U(\mathbf{n})$, $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^r$ with $\mathbf{m} - \mathbf{n} \in \Lambda(\psi)$ as

$$\tilde{\gamma}(X \otimes \mathbf{t}^{\mathbf{m}-\mathbf{n}}) = \gamma(X \otimes \mathbf{t}^{\mathbf{m}-\mathbf{n}}) \otimes \mathbf{t}^{\mathbf{m}} = X \otimes \mathbf{t}^{\mathbf{m}}.$$

On the other hand, let X be a nonzero homogeneous element of $\text{Im}\tilde{\gamma}$ of degree $\mathbf{m} \in \mathbb{Z}^r$. As $X \in L_r(V^0(\psi, 0))$, we have $X = \sum_{i=1}^k X_i \otimes \mathbf{t}^{\mathbf{m}}$, where $X_i \in U(L_r(\mathfrak{g} \otimes t_0^{-1}\mathbb{C}[t_0^{-1}])) \subset V^0(\psi, 0)$ are nonzero homogeneous of distinct degrees \mathbf{n}_i . Since $X \in \text{Im}\tilde{\gamma}$, there exist nonzero homogeneous $Y_1, \dots, Y_\ell \in U(L_r(\mathfrak{g} \otimes t_0^{-1}\mathbb{C}[t_0^{-1}]))$ of distinct degrees \mathbf{p}_j with $\mathbf{m} - \mathbf{p}_j \in \Lambda(\psi)$ such that

$$\sum_{i=1}^k X_i \otimes \mathbf{t}^{\mathbf{m}} = \tilde{\gamma} \left(\sum_{j=1}^{\ell} Y_j \otimes \mathbf{t}^{\mathbf{m}-\mathbf{p}_j} \right) = \sum_{j=1}^{\ell} Y_j \otimes \mathbf{t}^{\mathbf{m}}.$$

Since X_1, \dots, X_k and Y_1, \dots, Y_ℓ are all homogeneous and nonzero, we must have $\{\mathbf{n}_i\}_{i=1}^k = \{\mathbf{p}_j\}_{j=1}^{\ell}$. Consequently, $\mathbf{m} - \mathbf{n}_i \in \Lambda(\psi)$. This proves that X is contained in the span. \square

We shall need the following technical result:

Lemma 6.8. *Let J be an ideal of $V(\psi, 0)$. Then*

$$\tilde{\gamma}(J) = (\text{Im } \tilde{\gamma}) \cap L_r(\gamma(J)). \quad (6.14)$$

Proof. From Corollary 3.7, J is automatically \mathbb{Z}^r -graded. From the definition of $\tilde{\gamma}$ we have $\tilde{\gamma}(J) \subset (\text{Im } \tilde{\gamma}) \cap L_r(\gamma(J))$. To prove the converse inclusion, let X be a nonzero homogeneous element in $(\text{Im } \tilde{\gamma}) \cap L_r(\gamma(J))$ of degree \mathbf{m} . Then $X = \sum_{i=1}^k X_i \otimes \mathbf{t}^{\mathbf{m}}$, where X_i are nonzero homogeneous elements in $U(L_r(\mathfrak{g} \otimes t_0^{-1}\mathbb{C}[t_0^{-1}]))$ of distinct degrees \mathbf{n}_i such that $\sum_{i=1}^k X_i \in \gamma(J)$. Let $Y \in J$ such that $\gamma(Y) = \sum_{i=1}^k X_i$. As J is \mathbb{Z}^r -graded, we have $Y = Y_1 + \cdots + Y_\ell$, where $Y_j \in J$ are nonzero homogeneous elements of distinct degrees \mathbf{p}_j . Furthermore, for each $1 \leq j \leq \ell$, we have

$$Y_j = \sum_{s=1}^{N_j} Y_{js} \otimes \mathbf{t}^{\mathbf{p}_j - \mathbf{m}_{js}},$$

where $Y_{js} \in U(L_r(\mathfrak{g} \otimes t_0^{-1}\mathbb{C}[t_0^{-1}]))$ are nonzero homogeneous elements of distinct degrees \mathbf{m}_{js} such that $\mathbf{p}_j - \mathbf{m}_{js} \in \Lambda(\psi)$ for all $1 \leq s \leq N_j$.

Let $U_{\mathbf{m}}$ be the subspace of $U(L_r(\mathfrak{g} \otimes t_0^{-1}\mathbb{C}[t_0^{-1}]))$, spanned by homogeneous elements of degree $\mathbf{n} \in \Lambda(\psi) + \mathbf{m}$, and let $U'_{\mathbf{m}}$ be the subspace spanned by homogeneous elements of degree $\mathbf{n} \notin \Lambda(\psi) + \mathbf{m}$. Then

$$U(L_r(\mathfrak{g} \otimes t_0^{-1}\mathbb{C}[t_0^{-1}])) = U_{\mathbf{m}} \oplus U'_{\mathbf{m}}.$$

Set

$$Y_{\mathbf{m}} = \sum_{\substack{1 \leq j \leq \ell \\ \mathbf{p}_j \in \Lambda(\psi) + \mathbf{m}}} Y_j \in J \quad \text{and} \quad Y'_{\mathbf{m}} = \sum_{\substack{1 \leq j \leq \ell \\ \mathbf{p}_j \notin \Lambda(\psi) + \mathbf{m}}} Y_j \in J.$$

Then $\gamma(Y_{\mathbf{m}}) \in U_{\mathbf{m}}$ and $\gamma(Y'_{\mathbf{m}}) \in U'_{\mathbf{m}}$. Since

$$\gamma(Y_{\mathbf{m}}) + \gamma(Y'_{\mathbf{m}}) = X_1 + \cdots + X_k \in U_{\mathbf{m}},$$

we get that $\gamma(Y'_{\mathbf{m}}) = 0$ and $\gamma(Y_{\mathbf{m}}) = X_1 + \cdots + X_k$. Thus, we may assume $Y = Y_{\mathbf{m}}$ in the first place. From the definition of $Y_{\mathbf{m}}$, we have $\mathbf{p}_j - \mathbf{m} \in \Lambda(\psi)$ for $1 \leq j \leq n$. Then $\mathbf{t}^{\mathbf{m} - \mathbf{p}_j} \in \text{Im } \psi \subset V(\psi, 0)$. As J is an ideal, we have $\sum_{j=1}^n \mathbf{t}^{\mathbf{m} - \mathbf{p}_j} \cdot Y_j \in J$ and

$$\tilde{\gamma} \left(\sum_{j=1}^n \mathbf{t}^{\mathbf{m} - \mathbf{p}_j} \cdot Y_j \right) = \gamma \left(\sum_{j=1}^n \mathbf{t}^{\mathbf{m} - \mathbf{p}_j} \cdot Y_j \right) \otimes \mathbf{t}^{\mathbf{m}} = \gamma(Y) \otimes \mathbf{t}^{\mathbf{m}} = \sum_{i=1}^k X_i \otimes \mathbf{t}^{\mathbf{m}}.$$

This proves $X = \sum_{i=1}^k X_i \otimes \mathbf{t}^{\mathbf{m}} \in \tilde{\gamma}(J)$. Thus $(\text{Im } \tilde{\gamma}) \cap L_r(\gamma(J)) \subset \tilde{\gamma}(J)$. Therefore, $(\text{Im } \tilde{\gamma}) \cap L_r(\gamma(J)) = \tilde{\gamma}(J)$. \square

Note that for $\mathbf{m} \in \mathbb{Z}^r$, if $\mathbf{t}^{\mathbf{m}} \in \text{Im}\psi$, then $\mathbf{t}^{-\mathbf{m}} \in \text{Im}\psi$. For $a \in \mathfrak{g}$, $\mathbf{n} \in \mathbb{Z}^r$, and $f(\mathbf{t}) = \sum_{\mathbf{m} \in \mathbb{Z}^r} \beta_{\mathbf{m}} \mathbf{t}^{\mathbf{m}} \in \text{Im}\psi$, set

$$X(a, \mathbf{n}, f(\mathbf{t})) = \sum_{\mathbf{m} \in \mathbb{Z}^r} \beta_{\mathbf{m}} a_{-1, \mathbf{m} + \mathbf{n}} (1 \otimes \mathbf{t}^{-\mathbf{m}}) \in V(\psi, 0). \quad (6.15)$$

Now, we are ready to give a characterization of $L(\psi, 0)$.

Proposition 6.9. *Set $\ell = \ell_1 + \cdots + \ell_s$. Then*

$$L(\psi, 0) = V(\psi, 0)/J,$$

where J is the ideal of $(r+1)$ -toroidal vertex algebra $V(\psi, 0)$, generated by

$$(e_{\theta}(-1, \mathbf{m}))^{\ell+1} \mathbf{1}, \quad X(a, \mathbf{n}, f(\mathbf{t})) \quad (6.16)$$

for $\mathbf{m} \in \mathbb{Z}^r$, $a \in \mathfrak{g}$, $\mathbf{n} \in \mathbb{Z}^r$, $f(\mathbf{t}) \in p_1(t_1)(\text{Im}\psi) + \cdots + p_r(t_r)(\text{Im}\psi) \subset L_r$.

Proof. For $a \in \mathfrak{g}$, $\mathbf{n} \in \mathbb{Z}^r$ and $f(\mathbf{t}) = \sum_{\mathbf{m} \in \mathbb{Z}^r} \beta_{\mathbf{m}} \mathbf{t}^{\mathbf{m}} \in \text{Im}\psi$, we have

$$\gamma(X(a, \mathbf{n}, f(\mathbf{t}))) = \sum_{\mathbf{m} \in \mathbb{Z}^r} \beta_{\mathbf{m}} (a \otimes t_0^{-1} \mathbf{t}^{\mathbf{m} + \mathbf{n}}) \mathbf{1} = (a \otimes t_0^{-1} \mathbf{t}^{\mathbf{n}} f(\mathbf{t})) \mathbf{1}.$$

Note that as $\text{Im}\psi$ is a nonzero \mathbb{Z}^r -graded subring of L_r , we have $(\text{Im}\psi)L_r = L_r$. Then $\mathbf{t}^{\mathbf{n}} p_j(t_j)(\text{Im}\psi)$ for $\mathbf{n} \in \mathbb{Z}^r$ linearly span $p_j(t_j)L_r$. Since the generators of J are homogeneous, J is simply the τ -submodule generated by those generators. It then follows that $\gamma(J) = I$.

Recall the commutative diagrams (6.7) and (6.8). As $L(\psi, 0)$ is graded simple, by Lemma 3.9, $\tilde{\pi}$ is injective. By Lemma 6.7, $\tilde{\gamma}$ is also injective. Then

$$\ker \alpha = \tilde{\gamma}^{-1}(\ker(\alpha^0 \otimes 1)) = \tilde{\gamma}^{-1}(L_r(I)).$$

As $\gamma(J) = I$, it can be readily seen that the ideal of vertex algebra $L_r(V^0(\psi, 0))$ generated by $\tilde{\gamma}(J)$ is equal to $L_r(I)$. By Lemma 6.8, we have $\tilde{\gamma}(J) = \text{Im}\tilde{\gamma} \cap L_r(I)$, which is equivalent to $J = \tilde{\gamma}^{-1}(L_r(I))$. Thus, $J = \tilde{\gamma}^{-1}(L_r(I)) = \ker \alpha$. Therefore, $V(\psi, 0)/J = L(\psi, 0)$. \square

Recall that a $V(\psi, 0)$ -module structure on a vector space amounts to a restricted τ -module structure with $\ker \psi$ acting trivially. We have:

Theorem 6.10. *An $L(\psi, 0)$ -module structure on a vector space W exactly amounts to a restricted and integrable τ -module structure such that $(\ker \psi)W = 0$ and*

$$p_j(x_j/t_j)a(x_0, \mathbf{x}) = 0 \quad \text{on } W \quad \text{for all } 1 \leq j \leq r, \quad a \in \mathfrak{g}. \quad (6.17)$$

Proof. Assume that W is an $L(\psi, 0)$ -module. We know that W is a restricted τ -module. From the definition of $L(\psi, 0)$, we get that $(\ker \psi)W = 0$. By Theorem 5.4, $L(\psi, 0)$ is an integrable τ -module. Then for $\alpha \in \Delta$, $e_\alpha \in \mathfrak{g}_\alpha$ and $\mathbf{m} \in \mathbb{Z}^r$, there exists a positive integer k such that $e_\alpha(-1, \mathbf{m})^k \mathbf{1} = 0$. Thus we have

$$e_\alpha(x_0, \mathbf{n})^k = Y_W^0(e_\alpha(-1, \mathbf{n})^k \mathbf{1}, x_0) = 0, \quad (6.18)$$

noticing that $[e_\alpha(m_0, \mathbf{n}), e_\alpha(n_0, \mathbf{n})] = 0$ for $m_0, n_0 \in \mathbb{Z}$. Let $w \in W$. As W is a restricted τ -module, there exists $q \in \mathbb{Z}$ such that $e_\alpha(n_0, \mathbf{n})w = 0$ for $n_0 > q$. Suppose that p is an integer such that $e_\alpha(n_0, \mathbf{n})$ are nilpotent on w for $n_0 \geq p$. Then $e_\alpha(x_0, \mathbf{n}) - \sum_{n_0=p}^q e_\alpha(n_0, \mathbf{n})x_0^{-n_0-1}$ is nilpotent on w . It follows that $e_\alpha(p-1, \mathbf{n})$ is nilpotent on w . By induction on p , we get that $e_\alpha(n_0, \mathbf{n})$ are nilpotent on w for $n_0 \in \mathbb{Z}$. This shows that W is integrable. Note that

$$\sum_{\mathbf{n} \in \mathbb{Z}^r} Y_W^0(X(a, \mathbf{n}, f(\mathbf{t}))\mathbf{1}, x_0)\mathbf{x}^{-\mathbf{n}} = f(\mathbf{x}/\mathbf{t})a(x_0, \mathbf{x}) \quad (6.19)$$

for $f(\mathbf{t}) \in p_1(t_1)(\text{Im}\psi) + \cdots + p_r(t_r)(\text{Im}\psi)$. Then from Proposition 6.9 we see that (6.17) holds.

For the other direction, assume W is a restricted and integrable τ -module satisfying the very condition. First, W is naturally a module for $V(S_\mathbf{c}, 0)$. Since $(\ker \psi)W = 0$, W is a module for $V(\psi, 0)$. In view of Proposition 6.9, we only need to show that for each $\mathbf{n} \in \mathbb{Z}^r$,

$$e_\theta(x_0, \mathbf{n})^{\ell+1} = 0 \quad \text{on } W, \quad (6.20)$$

where $\ell = \ell_1 + \cdots + \ell_s$. For $\mathbf{n} \in \mathbb{Z}^r$, set

$$\mathfrak{s}_\mathbf{n} = \mathfrak{g}_\theta \otimes \mathbf{t}^\mathbf{n} \mathbb{C} [t_0^{\pm 1}] + \mathfrak{g}_{-\theta} \otimes \mathbf{t}^{-\mathbf{n}} \mathbb{C} [t_0^{\pm 1}] + \mathbb{C} \check{\theta} \otimes \mathbb{C} [t_0^{\pm 1}] + \mathbb{C} \mathbf{c}.$$

Notice that $\mathfrak{s}_\mathbf{n}$ is a subalgebra of $L_r(\widehat{\mathfrak{g}})$, which is isomorphic to $\widehat{\mathfrak{sl}}_2$. Since $\psi(\ell - \mathbf{c}) = \ell - \sum_{i=1}^s \ell_i = 0$ and $(\ker \psi)W = 0$, \mathbf{c} acts on W as scalar ℓ . As a restricted and integrable $\mathfrak{s}_\mathbf{n}$ -module, from [DLM] W is a direct sum of irreducible highest weight integrable modules of level ℓ . Then (see [LP]) we have (6.20). Therefore, W is an $L(\psi, 0)$ -module. \square

Recall that $\tilde{\gamma}$ is an $(r+1)$ -toroidal vertex algebra homomorphism from $V(\psi, 0)$ to $L_r(V^0(\psi, 0))$. Through $\tilde{\gamma}$, every $L_r(V^0(\psi, 0))$ -module is naturally a $V(\psi, 0)$ -module. On the other hand, we have:

Lemma 6.11. *Let (W, Y_W) be a $V(\psi, 0)$ -module. Suppose that (W, ρ) is also an L_r -module such that*

$$Y_W(\mathbf{t}^\mathbf{m}; x_0, \mathbf{x}) = \rho(\mathbf{t}^\mathbf{m}) \quad \text{and} \quad Y_W(v; x_0, \mathbf{x})\rho(\mathbf{t}^\mathbf{n}) = \rho(\mathbf{t}^\mathbf{n})Y_W(v; x_0, \mathbf{x})$$

for $v \in V(\psi, 0)$, $\mathbf{m} \in \Lambda(\psi)$ and $\mathbf{n} \in \mathbb{Z}^r$. Then there exists an $L_r(V^0(\psi, 0))$ -module structure Y'_W on W such that

$$Y'_W(\tilde{\gamma}(v); x_0, \mathbf{x}) = Y_W(v; x_0, \mathbf{x}) \quad \text{for } v \in V(\psi, 0).$$

Proof. Recall from Proposition 3.6 that $V(\psi, 0)$ and $L_r(V^0(\psi, 0))$ are naturally \mathbb{Z}^r -graded vertex algebras. Denote by $Y^{(1)}$ and $Y^{(2)}$ the vertex operator maps of $V(\psi, 0)$ and $L_r(V^0(\psi, 0))$, respectively. From Theorem 2.12, W is a module for $V(\psi, 0)$ viewed as a vertex algebra. Denote by $Y_W^{(1)}$ the $V(\psi, 0)$ -module structure on W . Recall that

$$V(\psi, 0) = U(L_r(\mathfrak{g} \otimes t_0^{-1}\mathbb{C}[t_0^{-1}])) \otimes \text{Im}\psi \quad \text{and} \quad V^0(\psi, 0) = U(L_r(\mathfrak{g} \otimes t_0^{-1}\mathbb{C}[t_0^{-1}]))$$

as vector spaces. Then $V^0(\psi, 0)$ is a \mathbb{Z}^r -graded vector space. For $\mathbf{m} \in \mathbb{Z}^r$, let $U(\mathbf{m})$ denote the homogeneous subspace of $V^0(\psi, 0)$ of degree \mathbf{m} . Define a linear map $Y_W^{(2)}(\cdot, x_0) : L_r(V^0(\psi, 0)) \rightarrow \text{Hom}(W, W((x_0)))$ by

$$Y_W^{(2)}(u \otimes \mathbf{t}^{\mathbf{n}}, x_0) = \rho(\mathbf{t}^{\mathbf{n}-\mathbf{m}})Y_W^{(1)}(u, x_0) \quad \text{for } u \in U(\mathbf{m}), \mathbf{m}, \mathbf{n} \in \mathbb{Z}^r.$$

From definition, for $u \in U(\mathbf{m})$, $\mathbf{n} \in \Lambda(\psi)$, we have

$$Y_W^{(2)}(\tilde{\gamma}(u \otimes \mathbf{t}^{\mathbf{n}}), x_0) = Y_W^{(2)}(u \otimes \mathbf{t}^{\mathbf{m}+\mathbf{n}}, x_0) = \rho(\mathbf{t}^{\mathbf{n}})Y_W^{(1)}(u, x_0) = Y_W^{(1)}(u \otimes \mathbf{t}^{\mathbf{n}}, x_0).$$

That is,

$$Y_W^{(2)}(\tilde{\gamma}(v), x_0) = Y_W^{(1)}(v, x_0) \quad \text{for } v \in V(\psi, 0).$$

We now show that $Y_W^{(2)}$ satisfies the Jacobi identity. Let $u \in U(\mathbf{m})$, $v \in U(\mathbf{n})$, $\mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q} \in \mathbb{Z}^r$. We have

$$\begin{aligned} & z^{-1}\delta\left(\frac{x-y}{z}\right)Y_W^{(2)}(u \otimes \mathbf{t}^{\mathbf{m}+\mathbf{p}}, x)Y_W^{(2)}(v \otimes \mathbf{t}^{\mathbf{n}+\mathbf{q}}, y) \\ & \quad - z^{-1}\delta\left(\frac{y-x}{-z}\right)Y_W^{(2)}(v \otimes \mathbf{t}^{\mathbf{n}+\mathbf{q}}, y)Y_W^{(2)}(u \otimes \mathbf{t}^{\mathbf{m}+\mathbf{p}}, x) \\ &= z^{-1}\delta\left(\frac{x-y}{z}\right)\rho(\mathbf{t}^{\mathbf{p}})Y_W^{(1)}(u, x)\rho(\mathbf{t}^{\mathbf{q}})Y_W^{(1)}(v, y) \\ & \quad - z^{-1}\delta\left(\frac{y-x}{-z}\right)\rho(\mathbf{t}^{\mathbf{q}})Y_W^{(1)}(v, y)\rho(\mathbf{t}^{\mathbf{p}})Y_W^{(1)}(u, x) \\ &= x^{-1}\delta\left(\frac{y+z}{x}\right)\rho(\mathbf{t}^{\mathbf{p}+\mathbf{q}})Y_W^{(1)}(Y^{(1)}(u, z)v, y). \end{aligned}$$

Note that

$$\begin{aligned} Y_W^{(1)}(Y^{(1)}(u, z)v, y) &= Y_W^{(2)}(\tilde{\gamma}(Y^{(1)}(u, z)v), y) = Y_W^{(2)}(Y^{(2)}(\tilde{\gamma}(u), z)\tilde{\gamma}(v), y) \\ &= Y_W^{(2)}(Y^{(2)}(u \otimes \mathbf{t}^{\mathbf{m}}, z)(v \otimes \mathbf{t}^{\mathbf{n}}), y) = Y_W^{(2)}(Y^{(2)}(u, z)v \otimes \mathbf{t}^{\mathbf{m}+\mathbf{n}}, y) \end{aligned}$$

and

$$\rho(\mathbf{t}^{\mathbf{n}})Y_W^{(2)}(u \otimes \mathbf{t}^{\mathbf{p}}, x) = \rho(\mathbf{t}^{\mathbf{n}+\mathbf{p}-\mathbf{m}})Y_W^{(1)}(u, x) = Y_W^{(2)}(u \otimes \mathbf{t}^{\mathbf{n}+\mathbf{p}}, x).$$

Then

$$\begin{aligned} \rho(\mathbf{t}^{\mathbf{p}+\mathbf{q}})Y_W^{(1)}(Y^{(1)}(u, z)v, y) &= \rho(\mathbf{t}^{\mathbf{p}+\mathbf{q}})Y_W^{(2)}(Y^{(2)}(u, z)v \otimes \mathbf{t}^{\mathbf{m}+\mathbf{n}}, y) \\ &= Y_W^{(2)}(Y^{(2)}(u, z)v \otimes \mathbf{t}^{\mathbf{m}+\mathbf{p}+\mathbf{n}+\mathbf{q}}, y) = Y_W^{(2)}(Y^{(2)}(u \otimes \mathbf{t}^{\mathbf{m}+\mathbf{p}}, z)(v \otimes \mathbf{t}^{\mathbf{n}+\mathbf{q}}), y). \end{aligned}$$

Now the Jacobi identity follows. Thus, $(W, Y_W^{(2)})$ is an $L_r(V^0(\psi, 0))$ -module. By Theorem 2.12, $Y_W^{(2)}$ gives rise to a module structure Y'_W on W for $L_r(V^0(\psi, 0))$ viewed as an $(r+1)$ -toroidal vertex algebra, as desired. \square

Furthermore, we have:

Proposition 6.12. *Let (W, Y_W) be any irreducible $V(\psi, 0)$ -module. Then there exists an (irreducible) $L_r(V^0(\psi, 0))$ -module structure Y'_W on W such that*

$$Y'_W(\tilde{\gamma}(v); x_0, \mathbf{x}) = Y_W(v; x_0, \mathbf{x}) \quad \text{for } v \in V(\psi, 0). \quad (6.21)$$

Proof. Recall that any $V(\psi, 0)$ -module is naturally an $\text{Im}\psi$ -module. Denote by ρ the representation map of $\text{Im}\psi$ on W . In view of Lemma 6.11, it suffices to prove that there exists a representation ρ' of L_r on W such that

$$\begin{aligned} \rho'(\mathbf{t}^{\mathbf{m}}) &= \rho(\mathbf{t}^{\mathbf{m}}) \quad \text{for } \mathbf{m} \in \Lambda(\psi), \\ Y_W(v; x_0, \mathbf{x})\rho'(\mathbf{t}^{\mathbf{n}}) &= \rho'(\mathbf{t}^{\mathbf{n}})Y_W(v; x_0, \mathbf{x}) \end{aligned}$$

for $v \in V(\psi, 0)$ and $\mathbf{n} \in \mathbb{Z}^r$. As an irreducible $V(\psi, 0)$ -module, W is naturally an irreducible τ -module. Since τ is of countable dimension (over \mathbb{C}), it follows that W is of countable dimension. As S_c lies in the center of $U(\tau)$, each element of S_c acts on W as a scalar. Consequently, each element of $\text{Im}\psi$ acts on W as a scalar. Since $\Lambda(\psi)$ is a cofinite subgroup of \mathbb{Z}^r (see Corollary 6.2), there exist $\mathbf{m}_1, \dots, \mathbf{m}_r \in \mathbb{Z}^r$ and positive integers d_1, \dots, d_r such that

$$\begin{aligned} \mathbb{Z}^r &= \mathbb{Z}\mathbf{m}_1 \oplus \dots \oplus \mathbb{Z}\mathbf{m}_r, \\ \Lambda(\psi) &= d_1\mathbb{Z}\mathbf{m}_1 \oplus \dots \oplus d_r\mathbb{Z}\mathbf{m}_r. \end{aligned}$$

For $1 \leq i \leq r$, assume that $\mathbf{t}^{d_i\mathbf{m}_i}$ acts as scalar c_i on W and choose a d_i -th root $(c_i)^{\frac{1}{d_i}}$ of c_i . We then define an L_r -module structure on W by

$$\mathbf{t}^{\mathbf{m}_i} \cdot w = (c_i)^{\frac{1}{d_i}} w \quad \text{for } 1 \leq i \leq r, \quad w \in W.$$

This action of L_r clearly extends that of $\text{Im}\psi$. The other assertion is also clear. \square

Recall that $\tilde{\pi}$ is an $(r+1)$ -toroidal vertex algebra homomorphism from $L(\psi, 0)$ to $L_r(L^0(\psi, 0))$. We have the following analogue of Proposition 6.12:

Theorem 6.13. *Let (W, Y_W) be any irreducible $L(\psi, 0)$ -module. Then there exists an irreducible $L_r(L^0(\psi, 0))$ -module structure Y''_W on W such that*

$$Y''_W(\tilde{\pi}(u); x_0, \mathbf{x}) = Y_W(u; x_0, \mathbf{x}) \quad \text{for } u \in L(\psi, 0).$$

Proof. Recall that $L^0(\psi, 0) = V^0(\psi, 0)/I$ and $L(\psi, 0) = V(\psi, 0)/J$. Note that W is naturally an irreducible $V(\psi, 0)$ -module. By Proposition 6.12, there exists an $L_r(V^0(\psi, 0))$ -module structure Y'_W on W satisfying (6.21). From the proof of Proposition 6.9, we see that $L_r(I)$ is generated by $\tilde{\gamma}(J)$. It follows that Y'_W gives rise to an $L_r(L^0(\psi, 0))$ -module structure Y''_W on W such that

$$Y''_W(\tilde{\pi}(v); x_0, \mathbf{x}) = Y_W(v; x_0, \mathbf{x}) \quad \text{for } v \in L(\psi, 0).$$

Since W is an irreducible $L(\psi, 0)$ -module, (W, Y''_W) must be also irreducible. \square

Note that Theorem 6.13 gives a classification of irreducible $L(\psi, 0)$ -modules in terms of irreducible $L_r(L^0(\psi, 0))$ -modules, while Propositions 3.12 and 4.28 give a classification of irreducible $L_r(L^0(\psi, 0))$ -modules in terms of integrable highest weight $\hat{\mathfrak{g}}$ -modules. On the other hand, using Theorems 6.10 and 6.13 we immediately have:

Corollary 6.14. *Let W be an irreducible restricted and integrable τ -module satisfying the conditions that $(\ker \psi)W = 0$ and that for all $1 \leq j \leq r$, $a \in \mathfrak{g}$,*

$$p_j(x_j/t_j)a(x_0, \mathbf{x}) = 0 \quad \text{on } W.$$

Then W is isomorphic to a τ -module of the form $(W_1 \otimes \cdots \otimes W_s)_{\mathbf{c}}$, where W_i are integrable highest weight $\hat{\mathfrak{g}}$ -modules of level ℓ_i ($1 \leq i \leq s$), $\mathbf{c} \in (\mathbb{C}^\times)^r$, and

$$(u \otimes \mathbf{t}^{\mathbf{m}}) \cdot (w_1 \otimes \cdots \otimes w_s) = \mathbf{c}^{\mathbf{m}} \sum_{i=1}^s \mathbf{a}_i^{\mathbf{m}} (w_1 \otimes \cdots \otimes uw_i \otimes \cdots \otimes w_s)$$

for $u \in \hat{\mathfrak{g}}$, $\mathbf{m} \in \mathbb{Z}^r$, $w_i \in W_i$ with $1 \leq i \leq s$.

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